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Intertemporal Allocation with a Non-convex Technology: The Aggregative Framework*

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1. INTRODUCTION

Convexity assumptions have played a dominant role in the development of the static Walrasian equilibrium theory with a finite number of agents and of the normative theories of productive efficiency and Pareto optimality.

In the literature on extending the static theory to an intertemporal framework, convex structures imposed on the technology and preferences are very often exploited to the fullest possible extent. In attempts to characterize efficient or optimal allocations in terms of the "dual" or "competitive conditions" of utility and profit maximization the relevant "prices" emerge out of an application of separation theorems on convex sets in finite or infinite dimensional linear spaces.

The duality theory has played an essential role in several contexts, and we mention a few of the important ones:

(a) in proving the existence of optimal programs, when future utilities are not discounted; [Gale (1967) developed his own version of the Kuhn–Tucker theorem applicable to maximization of a concave function over a convex set.]

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(b) in establishing "turnpike" or asymptotic behavior of optimal programs when the criterion of optimality is either a welfare function on terminal stocks [as in Radner (1961)] or the sum of additively separable utility functions defined on consumptions in each period [as in McKenzie (1976)];

(c) in a rigorous development of the concept of present-value maximization as an investment criterion and the notion of interest rates arising out of present value prices [Koopmans (1958); Malinvaud (1953), Radner (1967)];

(d) in the theory of decentralization of decision making in an informationally efficient manner [Hurwicz (1973)].

Quite apart from the development of the duality theory, it should be noted that the "standard" arguments on uniqueness and sensitivity of optimal programs rest directly on convexity of the technology. Thus, it is fair to single out convexity as perhaps the most crucial building block of the conventional models of intertemporal allocation. Hence, a major criticism of these models arises out of the inability to deal with indivisibilities or increasing resurns. To take just one example, John Hicks (1960) in his review of the progress of capital theory remarked, "I find it hard to believe that increasing returns and growth by capital accumulation are not tied very closely together. I could quote authority (Adam Smith or Allyn Young) for this belief."

While the recognition of the relevance of increasing returns is prevalent in the literature, rigorous analysis of models with non-convexities has so far been limited to a few notable efforts, mostly in the static context. [See, for example, Starr (1969), Arrow and Hurwicz (1977), Heal (1971), Brown and Heal (1979), Calsamiglia (1977), Guesnerie (1974).] It should be noted that a remarkable theorem due to Liapunov enables one to make progress in models with a continuum of agents without any convexity assumptions on individuals (see, for example, Hildenbrand (1974)].

In this paper, we consider the standard one good model, and in contrast with the literature growing out of Ramsey (1928), our gross output function exhibits an initial phase of increasing returns, with decreasing returns setting in eventually [see Fig. 1 in Section 2]. We reexamine some of the "classical" questions of intertemporal allocation theory, and attempt to derive definitive results using the special, recursive structure of the model.

First, we take up the question of characterizing resource allocation programs that are intertemporally efficient. We discover a basic qualitative difference between the "classical" model (see Section 2 for a clarification of this nomenclature) studied by Cass (1972) and others, and the "non-classical" model, that we study. It turns out that the average productivity, along a feasible program, has a crucial role in indicating intertemporal inefficiency in the "non-classical" model. Contrast this with the "classical" model in which the relevant signal can be precisely expressed solely in terms of marginal productivities.

Next, we consider the questions of existence and turnpike properties of programs that are optimal (according to a version of the overtaking criterion) when future utilities are not discounted. [The utility function is assumed to be concave.] It is perhaps a bit surprising that the special structure of the technology can be exploited and a fairly complete theory can be developed in this case, with suitable adaptations of a number of standard arguments. First, we show that there is a unique stationary program which maximizes utility per period among all stationary programs. This program can be supported by a stationary competitive price system. A "value loss" property relative to this stationary program, at this price-system can be derived; and that property is then used to show that this stationary program is optimal among all programs (stationary or not) from the same initial stock. [That is, this program is an optimal stationary program.] Finally, it is shown that an optimal program exists from every positive initial stock, and does indeed converge to the optimal stationary program.

When future utilities are discounted, the difference between the classical and non-classical models turns out to be quite remarkable. The qualitative properties of optimal programs depend crucially on the magnitude of the discount factor. Roughly speaking, when discounting is "mild," optimal programs behave as in the undiscounted case (converging to a unique optimal stationary program). When discounting is "heavy," optimal programs converge toward zero in the input, output and consumption levels. In the intermediate cases, optimal programs may exhibit a wide variety of behavior: in particular, in the interesting cases where there are two Euler stationary programs [see Section 2 for a definition of this concept], they may converge to the higher Euler stationary program, converge towards zero, or oscillate around the lower Euler stationary program.

We also note that the behavior of an optimal program [even in the longrun] is no longer independent of initial stocks. That is, optimal programs from different initial stocks can exhibit different asymptotic properties. Contrast this with the literature on 'turnpike" theory in the classical model, which establishes precisely that long-run optimal behavior is invariant with respect to initial conditions.

We have found very few results in the literature which overlap with ours. However, there are some papers which deal with problems and models similar to ours, and we mention a few of the important ones. Clark (1971) has examined the problem of a revenue maximizing fishery, when the "reproduction function" is a convex function for low input levels, and a concave function for high input levels. This is mathematically equivalent to the problem we study in Section 5, but with a linear utility function. Because of this added structure, Clark is able to rule out some types of behavior of optimal programs which we encounter for the non-linear utility function. A brief discussion of the discounted optimal growth problem with a linear utility function is given in Section 6. Lewis and Schmalense (1977) treat the case of a concave "reproduction function," but allow the utility function (in their discounted optimality exercise) to be non-concave; also, their treatment of the problem is in continuous time, in contrast to our discrete time methods.

Lane (1977) encounters a non-concave gross-output function in his continuous time treatment of a discounted optimal growth problem with endogenous population. In view of the fact that the consequent non-convexity in the technology set has less structure than ours, the results on the behavior of optimal programs are somewhat less sharp than ours. Finally, Skiba (1978) considers the problem of discounted optimality in continuous time in the framework of our model. Some of the observations of Skiba are rigorously established in our analysis. [His dynamic analysis often rests on a linear approximation technique, which is not quite an appropriate method for global analysis.] Furthermore, the effect of the discount rate and the initial stock on long-run optimal behavior is more systematically treated in this paper.

2. The Model

2a. *Production*

We consider an aggregative model, with a technology given by a function f from $R+$ to itself. The production possibilities consist of inputs, x, and outputs $y = f(x)$ for $x \geqslant 0$. The following assumptions on f are maintained throughout the paper.

(A.1) $f(0) = 0$.

(A.2) $f(x)$ is strictly increasing for $x \geqslant 0$.

(A.3) $f(x)$ is twice continuously differentiable for $x \geqslant 0$.

(A.4) f satisfies the following end-point conditions:

$$f'(\infty) < 1 < f'(0) < \infty$$

(A.5) There is a real number k_1 such that

(i) $0 < k_1 < \infty$;

(ii) $f''(x) = 0$ for $x = k_1$;

(iii) $f''(x) > 0$ for $0 \leqslant x < k_1$;

(iv) $f''(x) < 0$ for $x > k_1$.

In contrast to the present model, the traditional aggregative framework would replace (A.5) by

(A.5′) f is strictly concave for $x \geqslant 0$ ($f''(x) < 0$ for $x > 0$) while preserving (A.1)–(A.4). [In some versions, (A.3) and (A.4) would also be modified to allow $f'(0) = \infty$.] In discussions to follow, we will find it convenient to refer to a model with assumptions (A.1)–(A.4) and (A.5′) as "classical", and to a model with (A.1)–(A.5) as "non-classical."

We define a function, h [representing the average product function], as follows:

$$h(x) = [f(x)/x] \quad \text{for} \quad x > 0; \qquad h(0) = \lim_{x \to 0} [f(x)/x]. \tag{2.1}$$

Under (A.1)–(A.5), it is easily checked that $h(0) = f'(0)$.

Under (A.1)–(A.5), there exist uniquely determined real numbers $k^*, \bar{k}, k_2$ satisfying

(i) $0 < k_1 < k_2 < k^* < \bar{k} < \infty$;

(ii) $f'(k^*) = 1$;

(iii) $f(\bar{k}) = \bar{k}$;

(iv) $f'(k_2) = h(k_2)$.

Furthermore, for $0 \leqslant x < k^*$, $f'(x) > 1$; and for $x > k^*$, $f'(x) < 1$; for $0 < x < \bar{k}$, $x < f(x) < \bar{k}$, and for $x > \bar{k}$, $\bar{k} < f(x) < x$; for $0 < x < k_2$, $f'(x) > h(x)$, and for $x > k_2$, $f'(x) < h(x)$. Also note that for $0 \leqslant x < k_2$, $h(x)$ is increasing, and for $x > k_2$, $h(x)$ is decreasing; for $0 \leqslant x < k_1$, $f'(x)$ is increasing, and for $x > k_1$, $f'(x)$ is decreasing.

The functions, f, f', and h, together with the numbers k_1, k_2, k^* and $\bar{k}$ may be represented diagrammatically as follows, in Figs. 1a and b.

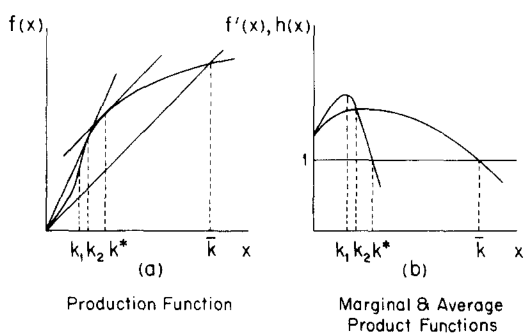

FIGURE 1

2b. *Programs*

A *feasible production program* from $\mathbf{x} > 0$ is a sequence $\langle x, y\rangle = \langle x_t, y_{t+1}\rangle$ satisfying

$$x_0 = \mathbf{x}; \qquad 0 \leqslant x_t \leqslant y_t \qquad \text{and} \qquad y_t = f(x_{t-1}) \qquad \text{for} \quad t \geqslant 1. \quad (2.2)$$

The *consumption program* $\langle c\rangle = \langle c_t\rangle$ generated by $\langle x, y\rangle$ is given by

$$c_t = y_t - x_t \qquad \text{for} \quad t \geqslant 1. \tag{2.3}$$

We will refer to $\langle x, y, c\rangle$ as a *feasible program* from $\mathbf{x}$, it being understood that $\langle x, y\rangle$ is a feasible production program, and $\langle c\rangle$ the corresponding consumption program.

A feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$ is called *positive* if $(x_t, y_{t+1}, c_{t+1}) \gg 0$ for $t \geqslant 0$. It is called *interior* if $\inf_{t\geqslant 0} x_t > 0$.

It is a standard exercise to check that for any feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$, we have $(x_t, y_{t+1}, c_{t+1}) \leqslant (\hat{k}, \hat{k}, \hat{k})$ for $t \geqslant 0$, where $\hat{k} = \max(\mathbf{x}, \bar{k})$.

A feasible program $\langle x', y', c'\rangle$ from $\mathbf{x}$ *dominates* a feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$, if $c'_t \geqslant c_t$ for all $t \geqslant 1$, and $c'_t > c_t$ for some t. A feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$, is said to be *inefficient* if some feasible program from $\mathbf{x}$ dominates it. It is said to be *efficient* if it is not inefficient.

2c. *Preferences*

The planner is endowed with a *utility function*, u, from $R+$ to R, and a *discount factor*, δ, where $0 < \delta \leqslant 1$, which reflects the planner's time preference. A feasible program $\langle x^*, y^*, c^*\rangle$ from $\mathbf{x}$ is called *optimal* if

$$\limsup_{T\to\infty} \sum_{t=1}^{T} \delta^{t-1}[u(c_t) - u(c_t^*)] \leqslant 0 \tag{2.4}$$

for every feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$.

A feasible program $\langle x^*, y^*, c^*\rangle$ from $\mathbf{x}$ is *intertemporal profit maximizing* (IPM) if there is a non-null sequence $\langle p^*\rangle = \langle p_t^*\rangle$ of non-negative prices, such that, for $t \geqslant 0$.

$$p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geqslant p_{t+1}^* y - p_t^* x \qquad \text{for} \quad x \geqslant 0,\ y = f(x). \tag{2.5}$$

A price sequence $\langle p^*\rangle = \langle p_t^*\rangle$ associated with an IPM program, for which (2.5) holds, is called a sequence of IPM prices. A feasible program $\langle x^*, y^*, c^*\rangle$ from $\mathbf{x}$ is *competitive* if there is a sequence $\langle p^*\rangle = \langle p_t^*\rangle$ of non-negative prices such that (2.5) holds for $t \geqslant 0$; and, for $t \geqslant 1$.

$$\delta^{t-1} u(c_t^*) - p_t^* c_t^* \geqslant \delta^{t-1} u(c) - p_t^* c, \qquad c \geqslant 0. \tag{2.6}$$

A price sequence $\langle p^*\rangle = \langle p_t^*\rangle$ associated with a competitive program

$\langle x^*, y^*, c^*\rangle$, for which (2.5), (2.6) hold, is called a sequence of *competitive prices*; (2.5), (2.6) are called the *competitive conditions*.

The following assumptions on u will be maintained throughout the paper:

(A.6) $u(c)$ is strictly increasing for $c \geqslant 0$.

(A.7) $u(c)$ is continuous for $c \geqslant 0$, and continuously differentiable for $c > 0$.

(A.8) $u(c)$ is strictly concave for $c \geqslant 0$, with $u''(c) < 0$ for $c > 0$.

(A.9) $u'(c) \to \infty$ as $c \to 0$.

A positive program $\langle x^*, y^*, c^*\rangle$ from $\mathbf{x}$ is called an *Euler program* if

$$u'(c_t^*) = \delta f'(x_t^*)\, u'(c_{t+1}^*) \qquad \text{for} \quad t \geqslant 1. \tag{2.7}$$

A feasible program $\langle x^*, y^*, c^*\rangle$ from $\mathbf{x}$ is *stationary* if $x_t^* = x_{t+1}^*$ for $t \geqslant 0$. An *Euler stationary program* (ESP) from $\mathbf{x}$ is a stationary program, which is also an Euler program. An *optimal stationary program* (OSP) from $\mathbf{x}$ is a stationary program, which is also an optimal program.

3. Characterization of Inefficiency

This section is devoted to finding suitable conditions, which can characterize the set of inefficient (alternatively, efficient) programs.

To provide such conditions, we have found it useful to look at the function $g(x)$ defined by

$$g(x) = \min[h(x), f'(x)] \qquad \text{for} \quad x \geqslant 0. \tag{3.1}$$

We associate, with any feasible program $\langle x, y, c\rangle$ from $\mathbf{x} > 0$, a sequence $\langle q_t\rangle$ given by

$$q_0 = 1, \qquad q_{t+1} = q_t / g(x_t) \qquad \text{for} \quad t \geqslant 0. \tag{3.2}$$

and a sequence $\langle r_t\rangle$ given by

$$r_0 = 1, \qquad r_{t+1} = [r_t / f'(x_t)] \qquad \text{for} \quad t \geqslant 0. \tag{3.3}$$

We establish that if a feasible program $\langle x, y, c\rangle$ is inefficient then $\langle(1/q_t)\rangle$ is summable [Theorem 3.1]. This result should be contrasted with the criterion of Cass (1972), established in a "classical" model, which says that if an interior program $\langle x, y, c\rangle$ is inefficient, then $\langle(1/r_t)\rangle$ is summable. That the Cass criterion does not hold in the "non-classical" model, is demonstrated in Example 3.1.

We then show that if an interior program $\langle x, y, c\rangle$ satisfies the condition that $\langle(1/r_t)\rangle$ is summable, then it is inefficient. It is not possible to show that "if $\langle(1/q_t)\rangle$ is summable, then $\langle x, y, c\rangle$ is inefficient." This is demonstrated in Example 3.2.

We also note, with an example, that efficient programs are not necessarily intertemporal profit maximizing, so that the well-known Malinvaud theory of the "classical" model breaks down [see Example 3.3]. Furthermore, it is not known in this "non-classical" framework, whether efficiency implies some concept of "value-maximization" relative to an appropriate "price system."

Before coming to the proofs of the theorems, we introduce some notation. Define $k_3 = (k_1 + k_2)/2$; $k_4 = (k_2 + k^*)/2$. The function $[-f''(x)]$ is continuous and positive on the compact set $[k_3, \bar{k}]$. Hence, there is $\mathbf{D} > 0$, such that $[-f''(x)] \geqslant \mathbf{D}$ for all x in $[k_3, \bar{k}]$. Similarly, $f'(x)$ is continuous and positive on the set $[0, \bar{k}]$. Hence there is $0 < \mathbf{B} < \infty$, such that $f'(x) \leqslant \mathbf{B}$ for all x in $[0, \bar{k}]$. Define

$$\mathbf{E} = \left[\frac{\mathbf{D}(k_4 - k_2)^2}{2\mathbf{B}\bar{k}^2}\right]; \qquad \text{then} \quad \mathbf{E} > 0.$$

Also, denote $\min[f'(k_4), h(0)]$ by $\mathbf{b}$; then $\mathbf{b} > 1$. Define $\mathbf{a} = (2\mathbf{b} - 1)/(\mathbf{b} - 1)$.

THEOREM 3.1. *If a feasible program $\langle x, y, c\rangle$ from $\mathbf{x} \in (0, \bar{k})$ is inefficient, then*

$$\sum_{t=0}^{\infty} (1/q_t) < \infty. \tag{3.4}$$

Proof. If $\langle x, y, c\rangle$ is inefficient, then by the argument in Cass (1972, pp. 203–204), there is a sequence $\langle \varepsilon_t\rangle$ and an integer $1 \leqslant s < \infty$, such that

$$\varepsilon_{t+1} = f(x_t) - f(x_t - \varepsilon_t) \qquad \text{for} \quad t \geqslant s, \tag{3.5}$$

$$0 < \varepsilon_t \leqslant x_t \leqslant \bar{k} \qquad \text{for} \quad t \geqslant s. \tag{3.6}$$

For $t \geqslant s$, define $x_t' = x_t - \varepsilon_t$. Then $0 \leqslant x_t' < x_t$, and $\varepsilon_{t+1} = f(x_t) - f(x_t')$ for $t \geqslant s$.

For each $t \geqslant s$, we distinguish between two cases:

Case I. $x_t' < k_3$.

Case II. $x_t' \geqslant k_3$.

We consider Case I first, and subdivide it into three subcases.

Case I(a). $x_t > k_4$.

Case I(b). $k_2 \leqslant x_t \leqslant k_4$.

Case I(c). $x_t < k_2$.

In case I(a), using (3.5) we get $\varepsilon_{t+1} = f(x_t) - f(x_t') = f(x_t) - f(k_2) + f(k_2) - f(x_t')$. Now, $f(x_t) - f(k_2) = f'(x_t)(x_t - k_2) + (1/2)[-f''(\xi_t)](x_t - k_2)^2$ [where $k_3 < k_2 \leqslant \xi_t \leqslant x_t \leqslant \bar{k}$] $\geqslant f'(x_t)(x_t - k_2) + (1/2)\mathbf{D}(x_t - k_2)^2 \geqslant f'(x_t)(x_t - k_2) + (1/2)\mathbf{D}(k_4 - k_2)^2 \geqslant f'(x_t)(x_t - k_2) + \frac{1}{2}(\mathbf{D}(k_4 - k_2)^2/\bar{k}^2)$. $\varepsilon_t^2 \geqslant f'(x_t)(x_t - k_2) + [\mathbf{D}(k_4 - k_2)^2/2\mathbf{B}\bar{k}^2]\, f'(x_t)\,\varepsilon_t^2 = f'(x_t)(x_t - k_2) + \mathbf{E}f'(x_t)\,\varepsilon_t^2$. Also, we note that $f(k_2) - f(x_t') = h(k_2)k_2 - h(x_t')x_t' \geqslant h(k_2)k_2 - h(k_2)x_t'$ [since $h(x)$ is non-decreasing for $0 \leqslant x \leqslant k_2$] $= h(k_2)(k_2 - x_t') = f'(k_2)(k_2 - x_t') \geqslant f'(x_t)(k_2 - x_t')$. Hence, $\varepsilon_{t+1} \geqslant f'(x_t)(x_t - k_2) + \mathbf{E}f'(x_t)\varepsilon_t^2 + f'(x_t)(k_2 - x_t') = f'(x_t)(x_t - x_t') + \mathbf{E}f'(x_t)\varepsilon_t^2 = f'(x_t)\varepsilon_t[1 + \mathbf{E}\varepsilon_t] = g(x_t)\varepsilon_t[1 + \mathbf{E}\varepsilon_t]$.

In case I(b), using (3.5), we get $\varepsilon_{t+1} = f(x_t) - f(x_t') = f(x_t) - f(k_2) + f(k_2) - f(x_t') \geqslant f'(x_t)(x_t - k_2) + f(k_2) - f(x_t')$. Using the arguments in case I(a), $f(k_2) - f(x_t') \geqslant f'(x_t)(k_2 - x_t')$. So $\varepsilon_{t+1} \geqslant f'(x_t)(x_t - k_2) + f'(x_t)(k_2 - x_t') = f'(x_t)\varepsilon_t = g(x_t)\varepsilon_t$, and $g(x_t) \geqslant f'(k_4) \geqslant \mathbf{b}$.

In case I(c), using (3.5), we get $\varepsilon_{t+1} = f(x_t) - f(x_t') = h(x_t)x_t - h(x_t')x_t' \geqslant h(x_t)(x_t - x_t')$ [h is non-decreasing for $0 \leqslant x \leqslant k_2$] $= g(x_t)(x_t - x_t') = g(x_t)\varepsilon_t$, and $g(x_t) \geqslant h(0) \geqslant \mathbf{b}$.

Now, we consider Case II. Here, by (3.6), $x_t > x_t' \geqslant k_3$. Hence, using (3.5), $\varepsilon_{t+1} = f(x_t) - f(x_t') = f'(x_t)(x_t - x_t') + \frac{1}{2}[-f''(\bar{\xi}_t)](x_t - x_t')^2$ [where $k_3 \leqslant \bar{\xi}_t \leqslant \bar{k}$] $\geqslant f'(x_t)\varepsilon_t + (\mathbf{D}/2)\varepsilon_t^2 \geqslant f'(x_t)\varepsilon_t + (\mathbf{D}/2\mathbf{B})f'(x_t)\varepsilon_t^2 \geqslant f'(x_t)\varepsilon_t + \mathbf{E}f'(x_t)\varepsilon_t^2 = f'(x_t)\varepsilon_t[1 + \mathbf{E}\varepsilon_t] \geqslant g(x_t)\varepsilon_t[1 + \mathbf{E}\varepsilon_t]$.

Thus in all cases we have either

$$\varepsilon_{t+1} \geqslant g(x_t)\varepsilon_t[1 + \mathbf{E}\varepsilon_t] \tag{3.7}$$

or

$$\varepsilon_{t+1} \geqslant g(x_t)\varepsilon_t \quad \text{and} \quad g(x_t) \geqslant \mathbf{b} > 1. \tag{3.8}$$

Let S be the set of time periods $t \geqslant s$ for which (3.7) holds; and, let S' be the set of time periods $t \geqslant s$, for which (3.7) does *not* hold.

First, S is an infinite set. For if S were finite, then there is $T < \infty$, such that for $t \geqslant T$, t is in S'. This means that for $t \geqslant T$, (3.8) holds. But then $\varepsilon_t \to \infty$ as $t \to \infty$, contradicting (3.6).

Now, there are two possibilities to consider: (i) S' is a finite set; (ii) S' is an infinite set. If (i) holds, there is $T' < \infty$, such that for $t \geqslant T'$, t is not in S'; that is, t is in S. This means that for $t \geqslant T'$, (3.7) holds. Then, by the argument of Cass (1972, p. 219), (3.4) holds.

In case (ii), S' is an infinite set, and, as we have already established, S is an infinite set. Now, define $\mu_t = \mathbf{E}$ when $t \in S$; $\mu_t = 0$ when $t \in S'$. Then, by (3.7), (3.8), we have for $t \geqslant s$,

$$\varepsilon_{t+1} \geqslant \varepsilon_t g(x_t)[1 + \mu_t\varepsilon_t] \tag{3.9}$$

This means $q_{t+1}\varepsilon_{t+1} \geqslant q_t\varepsilon_t[1+\mu_t\varepsilon_t]$ or

$$\frac{1}{q_{t+1}\varepsilon_{t+1}} \leqslant \frac{1}{q_t\varepsilon_t[1+\mu_t\varepsilon_t]} = \frac{1}{q_t\varepsilon_t} - \frac{\mu_t}{[q_t(1+\mu_t\varepsilon_t)]}$$

or

$$\frac{\mu_t}{q_t(1+\mu_t\varepsilon_t)} \leqslant \frac{1}{q_t\varepsilon_t} - \frac{1}{q_{t+1}\varepsilon_{t+1}},$$

so

$$\sum_{t=s}^{T} \frac{\mu_t}{q_t(1+\mu_t\varepsilon_t)} \leqslant \frac{1}{q_s\varepsilon_s} - \frac{1}{q_{T+1}\varepsilon_{T+1}} \leqslant \frac{1}{q_s\varepsilon_s}$$

for $T \geqslant s$. Hence, we have $\sum_{t=s}^{\infty} \mu_t/q_t(1+\mu_t\varepsilon_t) < \infty$. Using the definition of μ_t, we obtain

$$\mathbf{A} \equiv \sum_{t\in S} (1/q_t) < \infty. \tag{3.10}$$

Let t_n $(n = 1, 2,...,)$ be the time periods which belong to S. For each n, we have either (i) $t_{n+1} > t_n + 1$, or (ii) $t_{n+1} = t_n + 1$. In case (i), the time periods, t, such that $t_n < t < t_{n+1}$, belong to S'. For these t,

$$q_{t_{n+1}} \leqslant q_t/\mathbf{b}^{[t_{n+1}-t]} \qquad \text{(from (3.8))}.$$

So

$$\sum_{t_n<t<t_{n+1}} (1/q_t) \leqslant (1/q_{t_{n+1}}) \sum_{i=1}^{\infty} (1/\mathbf{b}^i) \leqslant [\mathbf{b}/(\mathbf{b}-1)][1/q_{t_{n+1}}].$$

Hence,

$$\sum_{t_n<t\leqslant t_{n+1}} (1/q_t) \leqslant \frac{(2\mathbf{b}-1)}{(\mathbf{b}-1)} (1/q_{t_{n+1}}) = (\mathbf{a}/q_{t_{n+1}}).$$

In case (ii),

$$\sum_{t_n<t\leqslant t_{n+1}} (1/q_t) = (1/q_{t_{n+1}}) \leqslant (\mathbf{a}/q_{t_{n+1}}).$$

Thus in either case,

$$\sum_{t_n<t\leqslant t_{n+1}} (1/q_t) \leqslant (\mathbf{a}/q_{t_{n+1}}). \tag{3.11}$$

Now pick any integer $N \geqslant 1$. Then

$$\sum_{t=t_1+1}^{t_{N+1}} (1/q_t) \leqslant \mathbf{a} \sum_{n=1}^{N} (1/q_{t_{n+1}}) \leqslant a\mathbf{A}.$$

Hence, we have $\sum_{t=t_1+1}^{\infty}(1/q_t) \leqslant \mathbf{aA}$. This means that (3.4) is satisfied. ∎

THEOREM 3.2. *An interior program $\langle x, y, c\rangle$ from $\mathbf{x} \in (0, \bar{k})$ is inefficient if*

$$\sum_{t=0}^{\infty} (1/r_t) < \infty \tag{3.12}$$

Proof. Follow exactly the method of Cass (1972, pp. 218–220), noting that concavity of f is nowhere required. ∎

Remarks. (1) Suppose a feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$ satisfies

$$\liminf_{t\to\infty} x_t > k^*,$$

then it is inefficient by Theorem 3.2.

(2) If $\mathbf{x} \geqslant \bar{k}$, then for a feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$, either (a) $x_t < \bar{k}$ after a finite number of periods; or (b) $x_t \geqslant \bar{k}$ for all $t \geqslant 0$. Clearly, in case (b), $\langle x, y, c\rangle$ is inefficient. Thus, there is no loss of generality in restricting $\mathbf{x}$ to be in $(0, \bar{k})$, as we have done in Theorems 3.1 and 3.2. We will continue this restriction in the subsequent sections.

(3) Cass (1972) establishes that, in a "classical" model, if an *interior* program $\langle x, y, c\rangle$ from $\mathbf{x} \in (0, \bar{k})$ is inefficient, then

$$\sum_{t=0}^{\infty} (1/r_t) < \infty. \tag{3.13}$$

Note that the method of proof used in Theorem 3.1, can be used in the "classical" model to show that if a *feasible* program $\langle x, y, c\rangle$ from $\mathbf{x} \in (0, \bar{k})$ is inefficient, then (3.13) holds. Thus the method of proof used in Theorem 3.1 is a refinement of the proof used in Cass (1972).

Given Theorems 3.1 and 3.2, a natural question is whether we can strengthen either of the Theorems to obtain a complete characterization of inefficiency. The answer is in the negative, as the following two examples demonstrate. The first example constructs an interior program $\langle x, y, c\rangle$ from $\mathbf{x} \in (0, \bar{k})$, which is inefficient, and violates (3.12). Hence the converse of Theorem 3.2 is not true. The second example constructs an interior program $\langle x, y, c\rangle$ from $\mathbf{x} \in (0, \bar{k})$, which satisfies (3.4) and is efficient. Hence, the converse of Theorem 3.1 is not true.

EXAMPLE 3.1. Define $H(x)=2x+x^2-\frac{1}{6}x^3$ for $0 \leqslant x \leqslant 5$. Then $H'(x)=2+2x-\frac{1}{2}x^2$, $H''(x)=2-x$. $H'(2+\sqrt{6}) = 2+2(2+\sqrt{6})-\frac{1}{2}(2+\sqrt{6})^2 = 2+4+2\sqrt{6}-\frac{1}{2}[4+4\sqrt{6}+6] = 2+4+2\sqrt{6}-2-2\sqrt{6}-3=1$. Also, $H''(x)>0$ for $x<2$; $H''(2)=0$; $H''(x)<0$ for $x>2$.

Clearly $H(0)=0$, $H(2)>6$. There exists $0<m<1$ such that $H(2-m) \geqslant 6$; then, $H''(x)=2-x \geqslant m$, for $x \leqslant 2-m$, $H'(2-m) \geqslant H'(0)=2$, $H'(2-m) \leqslant H'(2)=4$. Note that $H''(x)<0$ for $4 \leqslant x \leqslant 5$; $H'(4)=2$, $H'(4\frac{3}{4})=\frac{7}{32}<\frac{1}{4}$. Also H' is continuous for $4 \leqslant x \leqslant 4\frac{3}{4}$. Hence, there is $2+\sqrt{6}<Z_1<4\frac{3}{4}$ such that $H'(Z_1)=[1/H'(2-m)]$. Denote Z_1-1 by Z_2; Z_1+1 by Z_3. Choose $0<\beta<1$, such that $2(1-\beta)Z_3 \leqslant m/16$, and $H'(2-m)2^{(1-\beta)}<H'[2-(m/2)]$. Choose $\alpha=[H'(Z_1)/\beta]$.

Define $G(x)=H(Z_1)+\alpha(x-Z_2)^{\beta}-\alpha$ for $x \geqslant Z_2$. Then $G(Z_1)=H(Z_1)$; $G'(x)=\alpha\beta(x-Z_2)^{\beta-1}$; $G'(Z_1)=\alpha\beta=H'(Z_1)$. $G''(x)=\alpha\beta(\beta-1)(x-Z_2)^{\beta-2}<0$; for $Z_1 \leqslant x \leqslant Z_3$, $[-G''(x)]x/G'(x) = (1-\beta)x/(x-Z_2) = (1-\beta)x \leqslant (1-\beta)Z_3$. Now $G'(Z_3) = \alpha\beta/2^{(1-\beta)} = 1/H'(2-m)\,2^{(1-\beta)} > \{1/H'[2-(m/2)]\}$. Also, $G'(Z_3) = \alpha\beta/2^{(1-\beta)} < \alpha\beta = [1/H'(2-m)]$. Hence, there is $\hat{m}$, such that $(m/2)<\hat{m}<m$, for which $G'(Z_3) = [1/H'(2-\hat{m})]$. Denote $(2-\hat{m})$ by Z_4.

Define $f(x)=H(x)$ for $0 \leqslant x \leqslant Z_1$; $f(x)=G(x)$ for $Z_1<x$. Define a sequence $\langle x, y, c\rangle$ from $\mathbf{x}=Z_3$ by $x_t=Z_3$ for $t \in T=\{0, 2, 4,...,\}$, $x_t=Z_4$ for $t \in T'=\{1, 3, 5,...,\}$. Then, for $t \in T$, $f(x_t)-x_{t+1} = f(Z_3)-Z_4 > f(Z_1)-Z_4 > f(2)-Z_4 > 6-(2-\hat{m})>0$. For $t \in T'$, $f(x_t)-x_{t+1} = f(Z_4)-Z_3 > f(2-m)-Z_3 \geqslant 6-Z_3 = 6-(Z_1+1) = 5-Z_1>5-4\frac{3}{4}>0$. Hence $\langle x, y, c\rangle$ is a feasible program from $\mathbf{x}=Z_3$.

For $t \in T$, $t \geqslant 2$, $\prod_{s=0}^{t-1} f'(x_s)=[f'(Z_3)f'(2-\hat{m})]^{(t/2)}=1$. Hence, $\sum_{t=0}^{N}(1/r_t) \to \infty$ as $N \to \infty$. So (3.13) is violated. We will show, now, that $\langle x, y, c\rangle$ is inefficient.

Choose $d=\min(1/\hat{m}, \frac{1}{2})$. Define $d_0=d$, $d_1=d_0 f'(Z_3)\,[1+\hat{m}d/64]$, and $d_2=d_1 f'(Z_4)[1-\hat{m}d/64]$. Then, clearly $d_0 f'(Z_3)<d_1<d_0$; $d_2=d_0 f'(Z_3)f'(Z_4)[1-(\hat{m}d/64)^2]<d_0$.

Define $\varepsilon_0=d_0$; $\varepsilon_t=d_1$ for $t \in T'$, and $\varepsilon_t=d_2$ for $t \in T$. Clearly then $0<\varepsilon_t<x_t$ for $t \geqslant 0$. We will show that $\varepsilon_{t+1} \geqslant f(x_t)-f(x_t-\varepsilon_t)$ for $t \geqslant 0$. This is clearly true for $t=0$. To see this, write $f(x_0)-f(x_0-\varepsilon_0)= f'(Z_3)\varepsilon_0+\frac{1}{2}[-f''(\xi_0)]\,\varepsilon_0^2$ (where $x_0-\varepsilon_0 \leqslant \xi_0 \leqslant x_0$) $= f'(Z_3)\varepsilon_0[1+\frac{1}{2}([-f''(\xi_0)]Z_3\varepsilon_0/f'(Z_3)Z_3)]$. Now, $[-f''(\xi_0)] = \alpha\beta(1-\beta)/(\xi_0-Z_2)^{2-\beta} \leqslant \alpha\beta(1-\beta)/(Z_1-Z_2)^{2-\beta} = \alpha\beta(1-\beta)2^{(2-\beta)}/2^{(2-\beta)} \leqslant \alpha\beta(1-\beta)2^{2-\beta}/(Z_3-Z_2)^{2-\beta} = [-f''(Z_3)]\,2^{2-\beta}$. Hence, $\frac{1}{2}[-f''(\xi_0)]Z_3/f'(Z_3) \leqslant \frac{1}{2}Z_3[-f''(Z_3)]\,2^{2-\beta}/f'(Z_3) \leqslant \frac{1}{2}(1-\beta)Z_3 2^{2-\beta} < 2(1-\beta)Z_3 \leqslant m/16 \leqslant [\hat{m}/8]$. Also $[\varepsilon_0/Z_3]<(\varepsilon_0/4)$. So $f(x_0)-f(x_0-\varepsilon_0) \leqslant f'(Z_3)\varepsilon_0\,[1+\hat{m}\varepsilon_0/64] = d_0 f'(Z_3)[1+\hat{m}d/64]=d_1=\varepsilon_1$.

Suppose, now, that $\varepsilon_{t+1} \geqslant f(x_t)-f(x_t-\varepsilon_t)$ is true for $t=0, 1,..., N$. We will show that it is true for $t=N+1$. There are two cases to consider: (i)

$N \in T$, (ii) $N \in T'$. In case (i), $f(x_{N+1}) - f(x_{N+1} - \varepsilon_{N+1}) = f(Z_4) - f(Z_4 - d_1) = f'(Z_4)\,d_1 + \frac{1}{2}[-f''(\xi_{N+1})]\,d_1^2 = f'(Z_4)\,d_1 [1 - \frac{1}{2}f''(\xi_{N+1})Z_4 d_1/f'(Z_4)Z_4]$ (where $Z_4 - d_1 \leqslant \xi_{N+1} \leqslant Z_4$). Now, $f''(\xi_{N+1}) = 2 - \xi_{N+1} \geqslant \hat{m}$ [since $\xi_{N+1} \leqslant 2 - \hat{m}$], and $f'(Z_4) < 4$. So, $f''(\xi_{N+1})Z_4/f'(Z_4) \geqslant \hat{m}(2 - \hat{m})/4 \geqslant \hat{m}/4$. Also, $d_1/Z_4 > d_0 f'(Z_3)/2 > d_0/8 = d/8$. Hence, $f(Z_4) - f(Z_4 - \delta_1) \leqslant f'(Z_4)\,d_1\ [1 - \hat{m}\delta/64] = d_2 = \varepsilon_{N+2}$.

In case (ii), $f(x_{N+1}) - f(x_{N+1} - \varepsilon_{N+1}) = f(Z_3) - f(Z_3 - d_2) \leqslant d_2 f'(Z_3)[1 + \hat{m}d_2/64]$ [by using the same calculations as those used for $t = 0] < d_0 f'(Z_3)[1 + \hat{m}d_0/64]$ $\{$since $d_2 < d_0\} = d_0 f'(Z_3)[1 + \hat{m}d/64] = d_1 = \varepsilon_{N+2}$. This completes the induction argument. Hence $\varepsilon_{t+1} \geqslant f(x_t) - f(x_t - \varepsilon_t)$ for $t \geqslant 0$ and $0 < \varepsilon_t < x_t$ for $t \geqslant 0$. So, by Cass (1972, pp. 203–204), $\langle x, y, c\rangle$ is inefficient.

EXAMPLE 3.2. Consider any f satisfying (A.1)–(A.5). Let $\mathbf{x} = (k_1 + k_2)/2$. Let $x' = \frac{1}{4}k_1 + \frac{3}{4}k_2$. Then $k_1 < x' < \mathbf{x} < k_2$. Also $f(\mathbf{x}) - \mathbf{x} > f(x') - x'$. Hence there is $0 < \varepsilon < f(\mathbf{x}) - \mathbf{x}$, such that $f(\mathbf{x}) - \mathbf{x} - \varepsilon > f(x') - x'$.

Define a sequence $\langle \hat{x}_t\rangle$ by $\hat{x}_0 = \mathbf{x}$, $\hat{x}_{t+1} = \hat{x}_t + \varepsilon$ for $\hat{x}_t \leqslant k^*$; $\hat{x}_{t+1} = \hat{x}_t$ for $\hat{x}_t > k^*$. Then $\langle \hat{x}_t\rangle$ is a feasible program, and there is some period for which $\hat{x}_t > k^*$. Let N be the first period for which $\hat{x}_t > k^*$. Let $\alpha = \prod_{s=0}^{N-1} f'(x_s)$. Then $\alpha > 1$.

Choose $k^* < m < \hat{x}_N$, such that $\alpha f'(m) > 1$, and $f(m) - m > f(\mathbf{x}) - \mathbf{x}$ [Clearly, such a choice of m is possible, and $f'(m) < 1$]. Denote $f'(m)$ by β. Then $\alpha\beta > 1$. Let M be the smallest integer such that $\beta^M\alpha \leqslant 1$. Clearly $M \geqslant 2$ and finite. Then $\beta^{M-1}\alpha > 1$. So, there is $1 > \gamma \geqslant \beta$ such that $\gamma\beta^{M-1}\alpha = 1$. Clearly, there is $k^* < \bar{m} \leqslant m$ such that $f'(\bar{m}) = \gamma$.

Define a sequence $\langle x, y, c\rangle$ as follows:

$$
\begin{aligned}
x_t &= \hat{x}_t && \text{for} \quad t = 0,\dots, N-1,\\
x_t &= m && \text{for} \quad t = N,\dots, N+M-2,\\
x_t &= \bar{m} && \text{for} \quad t = N+M-1,\\
x_t &= x_{t-(N+M)} && \text{for} \quad t \geqslant N+M.
\end{aligned}
$$

It can be checked that $\langle x, y, c\rangle$ is a feasible program from $\mathbf{x}$ [it is also interior] with $c_t \geqslant f(\mathbf{x}) - \mathbf{x} - \varepsilon > f(x') - x' > 0$ for $t \geqslant 1$.

Denote $[1/q_{N+M}]$ by e, and $\sum_{t=1}^{N+M}(1/q_t)$ by e'. By construction $[1/r_{N+M}] = 1$, and $e < 1$.

Given any positive integer, n, $\sum_{t=1}^{n(N+M)}(1/r_t) \geqslant n$, so $\sum_{t=0}^{T}(1/r_t) \to \infty$ as $T \to \infty$. Also, given any positive integer, n,

$$\sum_{t=1}^{n(N+M)} (1/q_t) = e' + ee' + \cdots + e^{n-1}e'$$
$$= e'(1-e^n)/(1-e) \leqslant e'/(1-e).$$

Hence, $\sum_{t=0}^{\infty}(1/q_t) < \infty$.

We claim that $\langle x, y, c\rangle$ is an efficient program. Otherwise there is a feasible program $\langle x'', y'', c''\rangle$ from $\mathbf{x}$ which dominates $\langle x, y, c\rangle$. By the procedure used in Cass (1972, pp. 203–204), there is S, and a sequence $\langle \varepsilon_t\rangle$ such that $0 < \varepsilon_t = (x_t - x_t'')$, and $\varepsilon_{t+1} \geqslant f(x_t) - f(x_t - \varepsilon_t)$ for $t \geqslant S$. Now, since $c_t > f(x') - x'$ for all t, $c_t'' > f(x') - x'$ for all t, and $f(x) - x < f(x') - x'$ for $x < x'$, so $x_t'' \geqslant x'$ for $t \geqslant 0$ [otherwise, x_t'' would become negative for large t, a contradiction]. Hence, both x_t'', $x_t' \geqslant x'$ for $t \geqslant 0$; similarly both x_t'', $x_t \leqslant \bar{k}$ for $t \geqslant 0$.

Note that $f''(x) < 0$ for $x \in (k_1, \bar{k}]$. Since $x' > k_1$, so there is $\mathbf{D}' > 0$, such that $[-f''(x)] \geqslant \mathbf{D}'$ for $x \in [x', \bar{k}]$. Now follow exactly the method of Cass (1972, pp. 218–219) to get $\sum_{t=0}^{\infty}(1/r_t) < \infty$. However, we have shown that $\sum_{t=0}^{T}(1/r_t) \to \infty$ as $T \to \infty$. This contradiction establishes our claim that $\langle x, y, c\rangle$ is efficient.

EXAMPLE 3.3. This example shows that an efficient program need not be intertemporal profit maximizing. Let $\mathbf{x} = k_1$, and consider the sequence $\langle x, y, c\rangle$ given by $x_t = \mathbf{x}$ for $t \geqslant 0$. Clearly, $\langle x, y, c\rangle$ is a feasible program from $\mathbf{x}$, and by Theorem 3.1 it is efficient. We claim it is not IPM. If it were, then there is a non-null sequence $\langle p_t\rangle$ of non-negative prices, such that (2.5) holds. Let n be the first period for which $p_n > 0$. Since $x_n > 0$, so $p_{n+1} > 0$ [using $x = 0$, $y = f(0) = 0$ in (2.5)]. Then (2.5) implies, $p_{n+1}f'(k_1) = p_n$, and $p_{n+1}[f(k_1) - f'(k_1)k_1] \geqslant 0$, so $f(k_1)/k_1 \geqslant f'(k_1)$, a contradiction.

4. OPTIMAL GROWTH WHEN FUTURE UTILITIES ARE UNDISCOUNTED

In this section, we study the questions of existence and turnpike properties of optimal programs, when future utilities are undiscounted.

Many of the results of the "classical" model continue to hold: (a) There is a unique Euler stationary program, and this is also the (unique) optimal stationary program; this program is competitive at a stationary price sequence; (b) optimal programs exist from every positive initial input level; they converge monotonically to the optimal stationary program.

Some results of the 'classical" model fail to hold: (a′) In general, it is not known whether an optimal program from every initial stock is unique; (b′) Optimal programs are not necessarily competitive, and an example is given to confirm this fact.

In this section, and the next, for expositional convenience, we will discuss the existence and qualitative properties of Euler and Optimal *Stationary* Programs in a first subsection; *non*-stationary optimal programs will be examined in a second sub-section.

4a. *Stationary Programs*

Consider the set $\mathscr{C} = \{c : c = f(x) - x,\ 0 \leqslant x \leqslant \bar{k}\}$. Clearly $\mathscr{C}$ is compact. Hence, there is c^* in $\mathscr{C}$, such that $c \leqslant c^*$ for all c in $\mathscr{C}$. Since $0 < x < \bar{k}$ implies $f(x) - x > 0$, so $c^* > 0$. Associated with c^* is x^* such that $0 < x^* < \bar{k}$, and $f(x^*) - x^* = c^*$. Since x^* maximizes $[f(x) - x]$ over the set $\{x : 0 \leqslant x \leqslant \bar{k}\}$, and the maximum is attained at an interior point,

$$f'(x^*) = 1.$$

Since k^* is the unique non-negative solution to $f'(x) = 1$, so $x^* = k^*$, and k^* is the unique input level, which maximizes c over the set $\mathscr{C}$.

Consider the program from k^* given by $x_t^* = k^*$, $y_{t+1}^* = f(k^*)$, $c_{t+1}^* = f(k^*) - k^*$, for $t \geqslant 0$. Then $\langle k^*, f(k^*), f(k^*) - k^* \rangle$ is a feasible program from k^*. Clearly, it is stationary, and positive. Since $f'(k^*) = 1$, so it is an Euler Stationary Program. Since k^* is the unique non-negative solution to $f'(x) = 1$, so it is also the only Euler Stationary Program.

We show, next, that $\langle k^*, f(k^*), f(k^*) - k^* \rangle$ is an Optimal Stationary program from k^*. For this, we need two preliminary results. For the rest of this section we denote $f(k^*) - k^*$ by c^*.

LEMMA 4.1. *There is $p^* > 0$, such that*

$$u(c^*) - p^*c^* \geqslant u(c) - p^*c \qquad \textit{for} \quad c \geqslant 0; \tag{4.1}$$

$$p^*f(k^*) - p^*k^* \geqslant p^*f(x) - p^*x \qquad \textit{for} \quad x \geqslant 0. \tag{4.2}$$

Proof. Denote $u'(c^*)$ by p^*; then $p^* > 0$. By concavity of u, we have for $c \geqslant 0$, $u(c) - u(c^*) \leqslant u'(c^*)(c - c^*) = p^*(c - c^*)$. By transposing terms, (4.1) is verified.

By definition of k^*, $f(k^*) - k^* \geqslant f(x) - x$ for $0 \leqslant x \leqslant \bar{k}$. For $x > \bar{k}$, $f(k^*) - k^* > 0 > f(x) - x$. So for all $x \geqslant 0$, $f(k^*) - k^* \geqslant f(x) - x$. Multiplying this inequality by $p^* > 0$, yields (4.2).

LEMMA 4.2. *Given any $\theta > 0$, there is $\eta > 0$, such that for $x \geqslant 0$, $(k^* - x) \geqslant \theta$ implies $[p^*f(k^*) - p^*k^*] - [p^*f(x) - p^*x] \geqslant \eta$.*

Proof. Suppose, on the contrary, there is a sequence $\langle x_n \rangle$ such that $x_n \geqslant 0$, $(k^* - x_n) \geqslant \theta$, for $n = 1, 2, 3, \ldots$, but $[p^*f(k^*) - p^*k^*] - [p^*f(x_n) - p^*x_n] \to 0$ as $n \to \infty$. Clearly x_n is in $[0, k^*]$ for each n, so consider a subsequence of $\langle x_n \rangle$ converging to $\hat{x}$. Then $\hat{x}$ is in $[0, k^*]$, and by continuity of f, $[p^*f(k^*) - p^*k^*] = [p^*f(\hat{x}) - p^*\hat{x}]$. Hence $f(\hat{x}) - \hat{x} = f(k^*) - k^*$. Since $(k^* - x_n) \geqslant \theta$ for each n, so $(k^* - \hat{x}) \geqslant \theta$, and $\hat{x} < k^*$. So $f(k^*) - f(\hat{x}) = f'(z)(k^* - \hat{x})$ where $\hat{x} < z < k^*$. Then $f'(z) > 1$, so $f(k^*) - f(\hat{x}) > k^* - \hat{x}$, a contradiction. This establishes the result.

THEOREM 4.1. *The feasible program $\langle k^*, f(k^*), c^* \rangle$ is an optimal program from k^*.*

Proof. Suppose on the contrary that there is a feasible program $\langle x, y, c \rangle$ from k^*, a scaler $\alpha > 0$, and a sequence of periods T_n $(n = 1, 2, 3,...)$, such that

$$\sum_{t=1}^{T_n} [u(c_t) - u(c^*)] \geqslant \alpha \qquad \text{for all } n. \tag{4.3}$$

Using Lemma 4.1, we have for $t \geqslant 1$, $u(c_t) - u(c^*) \leqslant p^*(c_t - c^*) = p^*[f(x_{t-1}) - x_t] - p^*[f(k^*) - k^*] = [p^*f(x_{t-1}) - p^*x_{t-1}] + [p^*x_{t-1} - p^*x_t] - p^*[f(k^*) - k^*] \leqslant [p^*x_{t-1} - p^*x_t]$. Hence, for $T \geqslant 1$, we have:

$$\sum_{t=1}^{T} [u(c_t) - u(c^*)] \leqslant \sum_{t=1}^{T} [p^*x_{t-1} - p^*x_t] = p^*k^* - p^*x_T. \tag{4.4}$$

Hence for all n, we have, using (4.3), (4.4),

$$p^*(k^* - x_{T_n}) \geqslant \alpha. \tag{4.5}$$

This means that $(k^* - x_{T_n}) \geqslant (\alpha/p^*)$ for all n, so by Lemma 4.2, there is $\varepsilon > 0$, such that

$$[p^*f(k^*) - p^*k^*] \geqslant [p^*f(x_{T_n}) - p^*x_{T_n}] + \varepsilon \qquad \text{for all } n. \tag{4.6}$$

Using Lemma 4.1 again, and (4.6), we have for $t = T_n + 1$, $u(c_t) - u(c^*) \leqslant p^*(c_t - c^*) = p^*[f(x_{t-1}) - x_t] - p^*[f(k^*) - k^*] = [p^*f(x_{t-1}) - p^*x_{t-1}] - [p^*f(k^*) - p^*k^*] + [p^*x_{t-1} - p^*x_t] \leqslant [p^*x_{t-1} - p^*x_t] - \varepsilon$. And for $t \neq T_n + 1$, we have by our previous calculations, $u(c_t) - u(c^*) \leqslant [p^*x_{t-1} - p^*x_t]$. Hence, for all n,

$$\begin{aligned} \alpha \leqslant \sum_{t=1}^{T_n} [u(c_t) - u(c^*)] &\leqslant p^*(k^* - x_{T_n}) - (n-1)\varepsilon \\ &\leqslant p^*k^* - (n-1)\varepsilon. \end{aligned}$$

For n large, this is a contradiction. Hence, $\langle k^*, f(k^*), c^* \rangle$ is an OSP. ∎

Remarks. (i) Note that, by construction, $\langle k^*, f(k^*), c^* \rangle$ is a stationary program from k^*, which has the maximum stationary consumption (hence utility) among all stationary programs from arbitrary initial stocks.

(ii) The program $\langle k^*, f(k^*), c^*\rangle$ is the only OSP in this model, from a positive initial input. For if there were another, say $\langle x, y, c\rangle$ from $\mathbf{x} > 0$, then it would be a positive program, and an Euler program. But $\langle k^*, f(k^*), c^*\rangle$ is the only Euler Stationary Program, so $\langle x, y, c\rangle$ could not be an OSP.

4b. *Non-Stationary Programs*

We will show that:

(a) there is an optimal program from every $\mathbf{x} \in (0, \bar{k})$;

(b) every optimal program converges to the OSP, $\langle k^*, f(k^*), c^*\rangle$.

We call a feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$, *good* if there exists $M > -\infty$, such that

$$\sum_{t=1}^{T} [u(c_t) - u(c^*)] \geqslant M \qquad \text{for all} \quad T \geqslant 1.$$

It is *bad* if

$$\sum_{t=1}^{T} [u(c_t) - u(c^*)] \to -\infty \qquad \text{as} \quad T \to \infty.$$

LEMMA 4.3. *There exists a good program from every* $\mathbf{x} \in (0, \bar{k})$.

Proof. Consider two cases (i) $\mathbf{x} \geqslant k^*$; (ii) $\mathbf{x} < k^*$. In case (i), the sequence $\langle x, y, c\rangle$ given by $x_0 = \mathbf{x}$, $y_1 = f(\mathbf{x})$, $c_1 = f(\mathbf{x}) - k^*$, $x_t = k^*$, $y_{t+1} = f(k^*)$, $c_{t+1} = f(k^*) - k^*$ for $t \geqslant 2$, is a feasible program, which is good.

In case (ii), define $\Phi(x) = f(x) - x$. Then, for $0 \leqslant x \leqslant k^*$, $\Phi'(x) = f'(x) - 1 \geqslant 0$. So, for $\mathbf{x} \leqslant x \leqslant k^*$, $f(x) - x \geqslant f(\mathbf{x}) - \mathbf{x} > 0$. Pick a positive integer N, such that $0 < [k^* - \mathbf{x}]/N \leqslant (\frac{1}{2})[f(\mathbf{x}) - \mathbf{x}]$. Now define a sequence $\langle \hat{x}, \hat{y}, \hat{c}\rangle$ as follows: $x_0 = \mathbf{x}$, $x_{t+1} = x_t + \{[k^* - \mathbf{x}]/N\}$ for $t = 0, 1, \ldots, N-1$; $x_t = k^*$ for $t \geqslant N$; $y_{t+1} = f(x_t)$, and $c_{t+1} = y_{t+1} - x_{t+1}$ for $t \geqslant 0$. Note that for $t = 0, \ldots, N-1$, $c_{t+1} = [f(x_t) - x_t] + [x_t - x_{t+1}] \geqslant [f(\mathbf{x}) - \mathbf{x}] - [x^* - \mathbf{x}]/N > 0$, by choice of N. Also, $x_N = \mathbf{x} + N[(k^* - \mathbf{x})/N] = k^*$. Hence, for $t \geqslant N$, $c_{t+1} = f(k^*) - k^* = c^*$. Thus $\langle \hat{x}, \hat{y}, \hat{c}\rangle$ is a feasible program from $\mathbf{x}$. Denote $\sum_{t=1}^{N} [u(c_t) - u(c^*)]$ by M. Then, $M > -\infty$, and clearly,

$$\sum_{t=1}^{T} [u(c_t) - u(c^*)] \geqslant M \qquad \text{for all} \quad T \geqslant 1.$$

Hence $\langle \hat{x}, \hat{y}, \hat{c}\rangle$ is good. ∎

Given Lemmas 4.1–4.3, one can follow standard arguments of Gale (1967), and Brock (1970), to complete the proof of existence of an optimal program. We give the main steps in the argument, without proofs, but with the appropriate references.

LEMMA 4.4. *If a feasible program from* $\mathbf{x} > 0$ *is not good, it is bad.*

Proof. See Gale (1967).

LEMMA 4.5. *If a feasible program* $\langle x, y, c \rangle$ *from* $\mathbf{x}$ *is good, then*

$$(x_t, y_t, c_t) \rightarrow \langle k^*, f(k^*), c^* \rangle \qquad as \quad t \rightarrow \infty.$$

Proof. See Gale (1967), or Brock (Theorem 2, 1970).

THEOREM 4.2. *There exists an optimal program from every* $\mathbf{x} \in (0, \bar{k})$.

Proof. See Brock (Lemma 5, and Theorem 2, 1970).

Since there exists a good program, an optimal program (whose existence is established in Theorem 4.2) is necessarily good. Consequently, by Lemma 4.5, every optimal program $\langle x, y, c \rangle$ from $\mathbf{x} \in (0, \bar{k})$ has the property that $(x_t, y_t, c_t) \rightarrow (k^*, f(k^*), c^*)$ as $t \rightarrow \infty$. Furthermore, if $\mathbf{x} = x^*$, then $\langle k^*, f(k^*), c^* \rangle$ itself is an optimal program by Theorem 4.1. If $\mathbf{x} < k^*$, then an optimal program $\langle x, y, c \rangle$ has (x_t, y_{t+1}, c_{t+1}) monotonically increasing for all $t \geqslant 0$, and $(x_t, y_{t+1}, c_{t+1}) \leqslant \langle k^*, f(k^*), c^* \rangle$ for all $t \geqslant 0$. [This assertion, and the next, follow directly from the argument used in Theorem 5.5 of the next section, so we omit the proof here]. Similarly, if $\mathbf{x} > k^*$, then an optimal program $\langle x, y, c \rangle$ has (x_t, y_{t+1}, c_{t+1}) monotonically decreasing for all $t \geqslant 0$, and $(x_t, y_{t+1}, c_{t+1}) \geqslant \langle k^*, f(k^*), c^* \rangle$ for all $t \geqslant 0$.

We now briefly discuss the differences between the "classical" and "non-classical" models, in the analysis of the problem of undiscounted optimality. It is fairly easy to check (actually, we provide a formal argument in the proof of Theorem 5.4 in the next section) that for $\mathbf{x} \geqslant k_2$, every optimal program is unique. But for $\mathbf{x} < k_2$, it is not known whether the result is true; we believe it is not, and a concrete example would be helpful. Certainly, the standard argument, used in the "classical" model does not go through.

Optimal programs from $\mathbf{x} \geqslant k_2$ can be shown to be competitive; but those from $\mathbf{x} < k_2$ are, in general, not. Consider, for example, $\mathbf{x} > 0$, such that $\mathbf{x} < f^{-1}(k_1)$. Consider an optimal program $\langle x, y, c \rangle$ from $\mathbf{x}$. If it is competitive there is a sequence $\langle p_t^* \rangle$ of non-negative prices such that (2.5), (2.6) hold. Now, by (2.6), $p_t^* > 0$ for $t \geqslant 1$, and by (2.5), $p_0^* > 0$ also. Also, by (2.5), since $u'(0)=\infty$, $c_t > 0$ for $t \geqslant 1$, so $(x_t, y_{t+1}) \gg 0$ for $t \geqslant 0$. Using (2.5), then $p^*_{t+1} f'(x_t) = p^*_t$, so that we have

$$p^*_{t+1} f(x_t) - p^*_{t+1} f'(x_t) x_t \geqslant p^*_{t+1} f(x) - p^*_{t+1} f'(x_t) x \qquad \text{for} \quad x \geqslant 0$$

or

$$f(x_t) - f'(x_t) x_t \geqslant f(x) - f'(x_t) x \qquad \text{for} \quad x \geqslant 0.$$

Using $x=0$ in the above inequality, $[f(x_t)/x_t] \geqslant f'(x_t)$. Since $\mathbf{x} < f^{-1}(k_1)$, so $x_1 \leqslant f(\mathbf{x}) < k_1$, and $[f(x_1)/x_1] < f'(x_1)$, a contradiction.

A difference related to the above is the following: in the "classical" model, an Euler program $\langle x, y, c\rangle$, satisfying $x_t \to k^*$ as $t \to \infty$, is optimal. [This can be established by showing that the Euler program is competitive, and then completing the argument by using the results of Peleg (1972).] In the "non-classical" case the argument fails (for the same reasons as mentioned above) for $\mathbf{x} < k_2$, since Euler programs need not be competitive. Whether the *result* still holds is an open question.

It can be shown in the "classical" model, that if $\langle x^*, y^*, c^*\rangle$ is an optimal program then it is an Euler program, and $x_t \to k^*$ as $t \to \infty$. This result continues to hold in the "non-classical" framework. However, it can also be shown in the "classical" model, that if $\langle x, y, c\rangle$ is an optimal program, then it is a competitive program, and $x_t \to k^*$ as $t \to \infty$. This result fails to hold in the "non-classical" case (see, again, the example given above). The difference is that an optimal program can be shown to be an Euler program in either case, since, loosely speaking, this involves a "local maximization argument." But showing an optimal program to be competitive involves showing that a "global maximization" occurs at the program [in terms of "intertemporal profits" and "utility minus expenditure"], and in the presence of a non-convexity in the production set, this will not, in general, hold.

The differences noted here, of course, continue to obtain in the "discounted case" of the next section, so, we will not repeat these observations there.

5. Optimal Growth When Future Utilities Are Discounted

The contrast between the 'classical" model, and the "non-classical" model is more pronounced when one considers the "discounted case."

We note, right at the outset, that when the discount factor, δ, satisfies $0 < \delta < 1$, then the existence of an optimal program follows from a direct compactness argument as indicated in Majumdar (1975, Theorem 1). In fact, given any $\mathbf{x} \in (0, \bar{k})$, it can be shown that there is an optimal program $\langle x^*, y^*, c^* \rangle$ from $\mathbf{x}$, such that

$$\infty > \sum_{t=1}^{\infty} \delta^{t-1} u(c_t^*) \geqslant \sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$$

for every feasible program $\langle x, y, c \rangle$ from $\mathbf{x}$.

To facilitate discussion of the types of behavior that optimal programs can demonstrate we use some convenient definitions, and the following classification of the magnitudes of the discount factor.

Case 1. $f'(k_1) < (1/\delta)$.

Case 2. $f'(k_1) = (1/\delta)$.

Case 3. $f'(k_1) > (1/\delta) > f'(k_2)$.

Case 4. $f'(k_2) \geqslant (1/\delta) > f'(0)$.

Case 5. $f'(0) \geqslant (1/\delta)$.

A feasible program $\langle x, y, c \rangle$ from $\mathbf{x} \in (0, \bar{k})$ is *dissipative* if $x_t \to 0$ as $t \to \infty$. [In this case, it follows, of course, that $(c_t, y_t) \to (0, 0)$ as $t \to \infty$]. It *converges to an ESP*, if $x_t \to \hat{x}$ as $t \to \infty$, and $\langle x', y', c' \rangle$ given by $x_t' = \hat{x}$, for $t \geqslant 0$, is an Euler stationary program. It is *oscillatory around* $\hat{x}$ if there is a subsequence of periods for which $x_t \geqslant \hat{x}$, and a subsequence of periods for which $x_t \leqslant \hat{x}$.

We discuss cases 1 and 5 first, since the results here are clear-cut. In case 1, (where the future is discounted "heavily") there is no ESP, and so no OSP; every optimal program is dissipative. In case 5, (where the future is discounted "mildly"), there is a unique ESP, and this is also the unique OSP. Every optimal program converges to this ESP. It is easy to check [following our methods in cases 1 and 5] that in a "classical" model [with $f'(0) < \infty$], if $f'(0) \leqslant (1/\delta)$, then there is no ESP, and every optimal program is dissipative. And, if $f'(0) > (1/\delta)$, then there is a unique ESP (which is also the unique OSP) and every optimal program converges to this ESP. [In the literature on the "classical" model, one generally observes only the latter case, since, almost invariably, it is also assumed that $f'(0) = \infty$].

The intermediate cases bring out the basic differences between the classical and the non-classical models. In particular in the non-classical case the behavior (even in the long-run) of optimal programs is quite sensitive to (a) the magnitude of the discount factor, and, perhaps more important to (b) the initial stock from which optimal programs start (given a discount factor). Recall, from our above discussion, that in the classical case, given a discount factor, the long-run behavior of optimal programs is invariant to the initial stock of the economy.

In case 2, there is a unique ESP at k_1, but there is no stationary program which is competitive. The unique ESP need not be an OSP, and this is confirmed by a concrete example. Optimal programs from $\mathbf{x} \in (0, k_1)$ are dissipative; those from $\mathbf{x} \in [k_1, \bar{k})$ are either dissipative or converge to the ESP. We also provide an example in which optimal programs from all $\mathbf{x} \in (0, \bar{k})$ are dissipative.

In case 3, there are two ESPs, the "higher" denoted by K^*, the "lower" by k_* [Here K^* (resp. k_*) represents the stationary input level on the higher (resp. lower) ESP]. There is, however, no stationary program which is competitive. That neither ESP need be an OSP is confirmed by an example. Optimal programs from $\mathbf{x} \in (0, \bar{k})$ are either dissipative, or oscillatory around k_*, or converge to the higher ESP, K^*.

In case 4, there are again two ESPs, the higher denoted by K^*, the lower by k_*. The higher ESP is the (unique) stationary program which is competitive; it is also the unique OSP. The lower ESP need not be an OSP. Optimal programs from $\mathbf{x} \in [k_2, \bar{k})$ converge to the higher ESP. Those from $\mathbf{x} \in (0, k_2)$ may exhibit one of the three types of behavior described in case 3. We provide an example, in this case, to show that asymptotic behavior of optimal programs from different initial stocks can be different. We establish, in this example, that an optimal program from a 'sufficiently low" input level, $\mathbf{x}$, must remain bounded away from K^* in input levels x_t; while, an optimal program from K^*, has $x_t = K^*$ for all $t \geq 0$.

We start our analysis with an elementary result, which is useful for all subsequent discussions.

LEMMA 5.1. (i) *If* $\langle x^*, y^*, c^* \rangle$ *is an optimal program from* $\mathbf{x} \in (0, \bar{k})$, *then it is an Euler program,* (ii) *If* $\langle x, y, c \rangle$ *is a competitive program from* $\mathbf{x} \geq 0$, *then it is an Euler program, and* $[f(x_t)/x_t] \geq f'(x_t)$ *for* $t \geq 0$.

Proof. To prove (i), note that by (A.9), $c_t^* > 0$ for $t \geq 1$, so $(x_t^*, y_{t+1}^*) \gg 0$ for $t \geq 0$. For each $t \geq 1$, the expression $u[f(x_{t-1}^*) - x] + \delta u[f(x) - x_{t+1}^*]$ is maximized at $x = x_t^*$, among all $x \geq 0$, satisfying $f(x_{t-1}^*) \geq x$, and $f(x) \geq x_{t+1}^*$. Since the maximum is at the interior point, so $u'(c_t^*) = \delta u'(c_{t+1}^*) f'(x_t^*)$ for $t \geq 1$.

To prove (ii), note that by (2.6), $p_t > 0$ for $t \geq 1$, and by (2.5), $p_0 > 0$. Hence, by (2.6) $c_t > 0$ for $t \geq 1$, and $(x_t, y_{t+1}) \gg 0$ for $t \geq 0$. Then, using (2.5), $p_{t+1} f'(x_t) = p_t$ for $t \geq 0$; while, by (2.6), $\delta^{t-1} u'(c_t) = p_t$ for $t \geq 1$. Hence, for $t \geq 1$, $u'(c_t) = \delta u'(c_{t+1}) f'(x_t)$. So $\langle x, y, c \rangle$ is an Euler program. Also, using (2.5), $p_{t+1} f(x_t) - p_{t+1} f'(x_t) x_t \geq p_{t+1} f(x) - p_{t+1} f'(x_t) x$ for $t \geq 0$. So using $x = 0$ in the above inequality, $[f(x_t)/x_t] \geq f'(x_t)$ for $t \geq 0$. ∎

5a. *Stationary Programs*

We will examine the questions of existence and uniqueness of Euler stationary programs, optimal stationary programs, and stationary programs which are competitive, in each of the five cases.

Case 1. $[f'(k_1) < (1/\delta)]$.

Here, for all $x \geqslant 0$, $f'(x) \leqslant f'(k_1) < (1/\delta)$. Consequently, for any Euler program $\langle x, y, c\rangle$, $u'(c_t) = \delta f'(x_t)\, u'(c_{t+1}) < u'(c_{t+1})$; that is, $c_t > c_{t+1}$. So, there is no Euler stationary program, and by Lemma 5.1, no OSP, and no stationary program which is competitive.

Case 2. $[f'(k_1) = (1/\delta)]$.

Here, for $x \geqslant 0$, $x \neq k_1$, we have $f'(x) < (1/\delta)$. The sequence $\langle x, y, c\rangle$, given by $x_t = k_1$, $y_{t+1} = f(k_1)$, $c_{t+1} = f(k_1) - k_1$ for $t \geqslant 0$, is a feasible program which is stationary. Clearly, it is an Euler program, so an ESP exists. It is easy to check also that this is the unique ESP. Since $f'(k_1) > [f(k_1)/k_1]$, so by Lemma 5.1, there is no stationary program which is competitive. Clearly, the program $\langle x, y, c\rangle$ is the only candidate for an OSP. It is not known whether $\langle x, y, c\rangle$ can be an OSP. However, one can construct an example where $\langle x, y, c\rangle$ is *not* an OSP. To see this, note that $1 < f(k_1)/k_1 < f'(k_1)$, so $[f(k_1) - k_1]/(1-\delta) = f(k_1)[1 - (k_1/f(k_1))]/[1 - (1/f'(k_1))] < f(k_1)$. Choose $0 < \alpha < 1$, with α sufficiently close to 1, so that $\{[f(k_1) - k_1]^\alpha/(1-\delta)\} < f(k_1)^\alpha$. Now, choose $u(c) = c^\alpha$. Then, for the stationary program $\langle x, y, c\rangle$ from k_1, $\sum_{t=1}^{\infty} \delta^{t-1} u(c_t) = [f(k_1) - k_1]^\alpha/(1-\delta)$. And, for the program $\langle x', y', c'\rangle$ from k_1, given by $x'_0 = k_1$, $y'_1 = f(k_1) = c'_1$, $(x'_t, y'_{t+1}, c'_{t+1}) = (0, 0, 0)$ for $t \geqslant 1$, we have $\sum_{t=1}^{\infty} \delta^{t-1} u(c'_t) = f(k_1)^\alpha$. Since $[f(k_1) - k_1]^\alpha/(1-\delta) < f(k_1)^\alpha$, so $\langle x, y, c\rangle$ is not an OSP.

Case 3. $[f'(k_1) > (1-\delta) > f'(k_2)]$.

In this case, there are two positive solutions to the equation: $\delta f'(x) = 1$. We call these solutions k_* and K^*, with $k_* < k_1 < K^* < k_2$. Clearly then, there are exactly two ESPs given by $\langle x, y, c\rangle$ and $\langle x', y', c'\rangle$ as follows: $x_t = k_* y_{t+1} = f(k_*)$, $c_{t+1} = f(k_*) - k_*$ for $t \geqslant 0$; $x'_t = K^*$, $y'_{t+1} = f(K^*)$, $c'_{t+1} = f(K^*) - K^*$. Since $f'(k_*) > [f(k_*)/k_*]$, and $f'(K^*) > (f(K^*)/K^*)$, so by Lemma 5.1, there is no stationary program which is competitive.

As in case 2, it is not known under what additional conditions, $\langle x, y, c\rangle$ or $\langle x', y', c'\rangle$ could be OSPs. However, one can again construct examples where neither of these is an OSP. Since $1 < f(K^*)/K^* < f'(K^*)$, and $1 < f(k_*)/k_* < f'(k_*)$, so $[f(K^*) - K^*]/(1-\delta) = f(K^*)[1 - (K^*/f(K^*)]/[1 - (1/f'(K^*))] < f(K^*)$, and $[f(k_*) - k_*]/(1-\delta) = f(k_*)$ $[1 - (k_*/f(k_*))]/[1 - (1/f'(k_*))] < f(k_*)$. Choose $0 < \alpha < 1$, with α sufficiently close to 1, so that $[f(K^*) = K^*]^\alpha/(1-\delta) < f(K^*)^\alpha$, and

$[f(k_*)-k_*]^\alpha/(1-\delta)<f(k_*)^\alpha$. Choose $u(c)=c^\alpha$. Now, $\sum_{t=1}^{\infty}\delta^{t-1}u(c_t)=[f(K^*)-K^*]^\alpha/(1-\delta)$. Also, the program $\langle\hat{x},\hat{y},\hat{c}\rangle$ given by $\hat{x}_0=K^*$, $\hat{y}_1=f(K^*)=\hat{c}_1$, $(\hat{x}_t,\hat{y}_{t+1},\hat{c}_{t+1})=(0,0,0)$ for $t\geqslant 1$, is feasible from K^*, and $\sum_{t=1}^{\infty}\delta^{t-1}u(\hat{c}_t)=f(K^*)^\alpha$. Since $[f(K^*)-K^*]^\alpha/(1-\delta)<f(K^*)$, so $\langle x,y,c\rangle$ is not an OSP. For exactly similar reasons $\langle x',y',c'\rangle$ is not an OSP.

Case 4. $[f'(k_2)\geqslant(1/\delta)>f'(0)]$.

Here, as in case 3, there are two positive solutions to the equation: $\delta f'(x)=1$. Call them K^* and k_*, with $K^*>k_*$. Clearly $0<k_*<k_1<k_2\leqslant K^*$. There are, consequently, exactly two Euler stationary programs, $\langle x,y,c\rangle$ and $\langle x',y',c'\rangle$ given by: $x_t=K^*$, $y_{t+1}=f(K^*)$, $c_{t+1}=f(K^*)-K^*$ for $t\geqslant 0$; $x'_t=k_*, y'_{t+1}=f(k_*), c'_{t+1}=f(k_*)-k_*$ for $t\geqslant 0$. Now, $f'(k_*)>f(k_*)/k_*$, so by Lemma 5.1, $\langle x',y',c'\rangle$ is not a stationary competitive program. However, $\langle x,y,c\rangle$ is competitive. To see this, define $p_t=\delta^{t-1}u'(c_t)$ for $t\geqslant 1$, and $p_0=p_1f'(K^*)$. Then, by concavity of u, we have $\delta^{t-1}u(c)-\delta^{t-1}u(c_t)\leqslant\delta^{t-1}u'(c_t)(c-c_t)=p_t(c-c_t)$, which yields (2.6). Define $\varnothing(x)=[f(k_2)/k_2]x$ for $0\leqslant x\leqslant k_2$, and $\varnothing(x)=f(x)$ for $x\geqslant k_2$. Then $\varnothing(0)=0$; $\varnothing$ is an increasing, concave differentiable function for $x\geqslant 0$. Also $\varnothing(x)\geqslant f(x)$ for $x\geqslant 0$. Hence, for $x\geqslant 0$, we have $p_{t+1}[f(x)-f(x_t)]\leqslant p_{t+1}[\varnothing(x)-\varnothing(x_t)]\leqslant p_{t+1}\varnothing'(x_t)[x-x_t]=p_{t+1}f'(x_t)[x-x_t]$, since $\varnothing'(x_t)=\varnothing'(K^*)=f'(K^*)=f'(x_t)$ for $t\geqslant 0$. Hence, $p_{t+1}f(x)-p_{t+1}f'(x_t)x\leqslant p_{t+1}f(x_t)-p_{t+1}f'(x_t)x_t$. Using the fact that $p_{t+1}f'(x_t)=\delta^t u'(c_{t+1})f'(x_t)=\delta^{t-1}u'(c_{t+1})[\delta f'(x_t)]=\delta^{t-1}u'(c_t)[\delta f'(K^*)]=\delta^{t-1}u'(c_t)=p_t$ for $t\geqslant 1$, and $p_1f'(x_0)=p_1f'(K^*)=p_0$, we have $p_{t+1}f(x)-p_tx\leqslant p_{t+1}f(x_t)-p_tx_t$ for $t\geqslant 0$, which is (2.5).

Note that $\langle x,y,c\rangle$ is competitive at the above defined price sequence $\langle p_t\rangle$; also, $p_tx_t=\delta^{t-1}u'(c_t)x_t=\delta^{t-1}u'[f(K^*)-K^*]K^*$, so that $\lim_{t\to\infty}p_tx_t=0$. Hence, by a completely standard argument, $\langle x,y,c\rangle$ is an optimal program from K^*. So $\langle x,y,c\rangle$ is an OSP. The only other candidate for an OSP is $\langle x',y',c'\rangle$ [by Lemma 5.1]. As in case 3, one can construct a utility function, for which $\langle x',y',c'\rangle$ is not an OSP; but, general conditions under which $\langle x',y',c'\rangle$ can be an OSP are not known.

Case 5. $[(1/\delta)\leqslant f'(0)]$.

In this case there is a unique positive solution to the equation: $\delta f'(x)=1$. Call this K^*. Then, we have $k^*>K^*>k_2$. Consequently, there is a unique Euler stationary program $\langle x,y,c\rangle$ given by: $x_t=K^*$, $y_{t+1}=f(K^*)$, $c_{t+1}=f(K^*)-K^*$ for $t\geqslant 0$. [Note that even if $\delta f'(0)=1$, the "zero program" does not qualify as an ESP, since by definition, an ESP must be a positive program]. Following the analysis of case 4, it can be shown that $\langle x,y,c\rangle$ is a stationary competitive program [it is the only one, by Lemma 5.1], and also an OSP [again, it is the only one, by Lemma 5.1].

5b. *Non-stationary Programs*

We will examine some qualitative properties of optimal programs from $\mathbf{x} \in (0, \bar{k})$ [that is, whether some monotone movements in inputs, outputs, and consumption levels can be observed], and also attempt to determine the long-run (asymptotic or "turnpike") behavior of such programs.

To simplify our discussion of the five cases, we first prove some results which are useful in the analysis of all the cases.

LEMMA 5.2. *If* $\langle x, y, c \rangle$ *from* $\mathbf{x} \in (0, \bar{k})$ *is an Euler program and* (i) $\delta f'(x_t) \leqslant 1$ *for some* $t \geqslant 1$, *then* (ii) $c_t \geqslant c_{t+1}$ *for this* t. *If there is strict inequality in* (i), *then there is strict inequality in* (ii). *If* (i) *holds for some* $t \geqslant 1$, *and* (iii) $x_t \geqslant x_{t-1}$ *for this* t, *then* (iv) $x_{t+1} \geqslant x_t$ *for this* t. *If there is strict inequality in* (i) *or* (iii), *then there is strict inequality in* (iv).

Proof. Since $\langle x, y, c \rangle$ is an Euler program, so $u'(c_t) = \delta f'(x_t) u'(c_{t+1})$. By (i), $u'(c_t) \leqslant u'(c_{t+1})$, so $c_t \geqslant c_{t+1}$. The statement about the strict inequalities is obvious.

If (i) and (iii) hold, then by the above argument, $c_t \geqslant c_{t+1}$, so that $f(x_{t-1}) - x_t \geqslant f(x_t) - x_{t+1}$. Using (iii), we have $f(x_t) - x_t \geqslant f(x_{t-1}) - x_t \geqslant f(x_t) - x_{t+1}$, so that $x_{t+1} \geqslant x_t$. Again, the statement about the strict inequalities is easy to check. ∎

LEMMA 5.3. *If* $\langle x, y, c \rangle$ *from* $\mathbf{x} \in (0, \bar{k})$ *is an Euler program, and* (i) $\delta f'(x_t) \geqslant 1$ *for some* $t \geqslant 1$, *then* (ii) $c_{t+1} \geqslant c_t$ *for this* t. *If there is strict inequality in* (i), *then there is strict inequality in* (ii). *If* (i) *holds for some* $t \geqslant 1$, *and* (iii) $x_{t-1} \geqslant x_t$ *for this* t, *then* (iv) $x_t \geqslant x_{t+1}$ *for this* t. *If there is strictly inequality in* (i) *or* (iii), *then there is strict inequality in* (iv).

Proof. Follow the method used to prove Lemma 5.2. ∎

LEMMA 5.4. *If* $\langle x, y, c \rangle$ *from* $\mathbf{x} \in (0, \bar{k})$ *is an Euler program, and* $x_t \to \hat{x}$ *as* $t \to \infty$, *where* $0 < \hat{x} < \bar{k}$, *then* $\delta f'(\hat{x}) = 1$.

Proof. Since $x_t \to \hat{x}$ as $t \to \infty$, so $c_t \to f(\hat{x}) - \hat{x} \equiv \hat{c}$. Since $0 < \hat{x} < \bar{k}$, so $\hat{c} > 0$. Since $\langle x, y, c \rangle$ is an Euler program, so $u'(c_t) = \delta f'(x_t) u'(c_{t+1})$ for $t \geqslant 1$, and consequently $u'(\hat{c}) = \delta f'(\hat{x}) u'(\hat{c})$. Hence $\delta f'(\hat{x}) = 1$. ∎

LEMMA 5.5. *If* $\langle x, y, c \rangle$ *is an optimal program from* $\mathbf{x} \in (0, \bar{k})$, *and* $T \geqslant 0$ *then* (1) $c_{t+2} \geqslant c_{t+1}$ *for all* $t \geqslant T$, *implies* $x_{t+1} \geqslant x_t$ *for* $t \geqslant T$; (2) $c_{t+2} \leqslant c_{t+1}$ *for all* $t \geqslant T$, *implies* $x_{t+1} \leqslant x_t$ *for all* $t \geqslant T$.

Proof. To prove (1), note that $x_{t+1} \leqslant k^*$ for $t \geqslant T$. Otherwise, if $x_{t+1} > k^*$ for some $t \geqslant T$, then $u'(c_{t+2}) \delta f'(x_{t+1}) < u'(c_{t+2}) \leqslant u'(c_{t+1})$, which contradicts Lemma 5.1. Suppose now that for some $S \geqslant T$, $x_{S+1} < x_S$.

Then $x_{S+2} = f(x_{S+1}) - c_{S+2} < f(x_S) - c_{S+2} \leqslant f(x_S) - c_{S+1} = x_{S+1}$. Then $c_{S+2} = f(x_{S+1}) - x_{S+2} = f(x_{S+1}) - x_{S+1} + (x_{S+1} - x_{S+2}) = f(x_{S+1}) - x_{S+1} + \theta_1$. Then $x_{S+3} = f(x_{S+2}) - c_{S+3} = [f(x_{S+2}) - x_{S+2}] + [x_{S+2} - c_{S+3}] < [f(x_{S+1}) - x_{S+1}] + [x_{S+2} - c_{S+3}]$ {since $x_{S+2} < x_{S+1} < k^*$} $\leqslant f(x_{S+1}) - x_{S+1} + x_{S+2} - c_{S+2} = [f(x_{S+1}) - x_{S+1}] + x_{S+2} - [f(x_{S+1}) - x_{S+1}] - \theta_1 = x_{S+2} - \theta_1$. Repeating this step, $x_{t+1} \leqslant x_t - \theta_1$ for $t \geqslant S+1$. Then $x_t < 0$ for large t, a contradiction.

To prove (2), suppose on the contrary that $x_{S+1} > x_S$ for some $S \geqslant T$. Then $c_{S+1} = f(x_S) - x_{S+1} = f(x_S) - x_S + x_S - x_{S+1} = [f(x_S) - x_S] - \hat{\theta}_1$. So $c_{t+1} \leqslant c_{S+1} = [f(x_S) - x_S] - \hat{\theta}_1$, (where $\hat{\theta}_1 > 0$) for $t \geqslant S$. Construct a sequence $\langle x', y', c' \rangle$ from $\mathbf{x}$, with $x'_t = x_t$ for $t = 0,..., S$, $x'_t = x_s$ for $t > S$. Then $\langle x', y', c' \rangle$ is feasible program from $\mathbf{x}$, with $c'_{t+1} = c_{t+1}$ for $t = 0,..., S-1$; $c'_{t+1} = f(x_S) - x_S \geqslant c_{t+1} + \hat{\theta}_1$ for $t \geqslant S$. Hence, $\langle x, y, c \rangle$ is inefficient, a contradiction, which proves (2).

We now begin our analysis of non-stationary optimal programs in the five cases.

Case 1. $[f'(k_1) < (1/\delta)]$.

The behavior of any optimal program $\langle x, y, c \rangle$ from $\mathbf{x} \in (0, \bar{k})$ is summarized in the following result.

THEOREM 5.1. *If $\langle x, y, c \rangle$ is an optimal program from $\mathbf{x} \in (0, \bar{k})$, then* (i) *$c_t$ is monotonically decreasing with* $\lim_{t\to\infty} c_t = 0$; (ii) *$x_t$ is monotonically decreasing with* $\lim_{t\to\infty} x_t = 0$; (iii) *$y_t$ is monotonically decreasing with* $\lim_{t\to\infty} y_t = 0$.

Proof. Here $\delta f'(x) \leqslant \delta f'(k_1) < 1$ for $x \geqslant 0$. Since $\langle x, y, c \rangle$ is optimal, so it is an Euler program by Lemma 5.1. By Lemma 5.2, c_t is monotonically decreasing. We claim that x_t is monotonically decreasing. If not, then $x_S \geqslant x_{S-1}$ for some S. By Lemma 5.2, x_t is non-decreasing for $t \geqslant S-1$. Since x_t is bounded above by $\bar{k}$, it converges to some $\hat{x}$. Since $x_t \geqslant x_{S-1} > 0$, for $t \geqslant S-1$, so $\hat{x} > 0$. Also, $\hat{x} < \bar{k}$, otherwise $\langle x, y, c \rangle$ is inefficient, by Theorem 3.2. Hence, by Lemma 5.4., $\delta f'(\hat{x}) = 1$, a contradiction, which establishes our claim. Since $y_t = c_t + x_t$, so y_t is monotonically decreasing.

We claim $\lim_{t\to\infty} x_t = 0$. Otherwise $\lim_{t\to\infty} x_t = \tilde{x} > 0$. Since x_t is decreasing, so $\tilde{x} < \bar{k}$. Hence $\delta f'(\tilde{x}) = 1$, a contradiction. Hence, $\lim_{t\to\infty} x_t = 0$. Since $y_t = f(x_{t-1})$, so $\lim_{t\to\infty} y_t = 0$. Since $c_t \leqslant y_t$, so $\lim_{t\to\infty} c_t = 0$. ∎

Case 2. $[f'(k_1) = (1/\delta)]$.

The monotonicity properties of optimal programs is noted in the following result.

PROPOSITION 5.1. *An optimal program $\langle x, y, c \rangle$ from $\mathbf{x} \in (0, \bar{k})$ has the* *following properties*; (i) c_t *is monotonically non-increasing*; (ii) x_t *is monotonically non-increasing*; (iii) y_t *is monotonically non-increasing*.

Proof. Here $\delta f'(x) < 1$ for $x \neq k_1$. Since $\langle x, y, c\rangle$ is optimal, it is an Euler program by Lemma 5.1. By Lemma 5.2, (i) follows immediately. Then (ii) follows by Lemma 5.5. Now, (iii) follows from (i) and (ii). ∎

The asymptotic properties of optimal programs is given by the following theorem.

THEOREM 5.2. *If* $\langle x, y, c\rangle$ *is an optimal program from* $\mathbf{x} \in (0, \bar{k})$, *then* (i) $(x_t, y_t, c_t) \to (0, 0, 0)$ *as* $t \to \infty$, *if* $\mathbf{x} \in (0, k_1)$; (ii) (x_t, y_t, c_t) *either converges to* $(0, 0, 0)$ *or to* $(k_1, f(k_1), f(k_1) - k_1)$ *as* $t \to \infty$, *if* $\mathbf{x} \in [k_1, \bar{k})$.

Proof. If $\mathbf{x} \in (0, k_1)$, then by Proposition 5.1, x_t converges to some $\tilde{x} < k_1$. If $\tilde{x} > 0$, then by Lemma 5.4, $\delta f'(\tilde{x}) = 1$, a contradiction. So $\tilde{x} = 0$. Since $y_t = f(x_{t-1})$, so $y_t \to 0$ as $t \to \infty$. Since $0 \leqslant c_t \leqslant y_t$, so $c_t \to 0$ as $t \to \infty$.

If $\mathbf{x} \in [k_1, \bar{k})$, then by Proposition 5.1, x_t converges to $\tilde{x} < \bar{k}$. Either (a) $\tilde{x} = 0$, or (b) $\tilde{x} > 0$. If (a) holds, then $x_t \to 0$ as $t \to \infty$; $y_t = f(x_{t-1}) \to 0$ as $t \to \infty$; also $0 \leqslant c_t \leqslant y_t$, so $c_t \to 0$ as $t \to \infty$. If (b) holds, then by Lemma 5.4, $\delta f'(\tilde{x}) = 1$, so $x = k_1$. Hence, $x_t \to k_1$, $y_t \to f(k_1)$ and $c_t \to f(k_1) - k_1$ as $t \to \infty$. ∎

The characterization of the long-run behavior of an optimal program when $\mathbf{x} \in [k_1, \bar{k})$, is somewhat less sharp than is desirable. More precisely, it would be useful to have conditions on u and f, under which one could definitely say that $(x_t, y_t, c_t) \to (0, 0, 0)$ for $\mathbf{x} \in [k_1, \bar{k})$. It is simple to construct an example in which this is indeed the case.

As in Section 5a, note that $[f(k_1) - k_1]/(1 - \delta) < f(k_1)$. Choose $0 < \alpha < 1$, such that $[f(k_1) - k_1]^\alpha/(1 - \delta) < f(k_1)^\alpha$. Now choose $u(c) = c^\alpha$. Then, by Proposition 5.1, an optimal program $\langle x, y, c\rangle$ from k_1 must either satisfy (a) $x_t = k_1$, $y_{t+1} = f(k_1)$, $c_{t+1} = f(k_1) - k_1$ for $t \geqslant 0$, or it must satisfy (b) $(x_t, y_t, c_t) \to (0, 0, 0)$ as $t \to \infty$. By the analysis in Section 5a, (a) cannot happen; so (b) must occur.

We claim that if $\langle x, y, c\rangle$ is optimal from $\mathbf{x} \in (k_1, \bar{k})$, then $(x_t, y_t, c_t) \to (0, 0, 0)$ as $t \to \infty$. Otherwise, by Proposition 5.1, $(x_t, y_t, c_t) \to (k_1, f(k_1), f(k_1) - k_1)$ as $t \to \infty$. Choose $\beta > 0$, such that $[f(k_1) - k_1 + \beta]^\alpha/(1 - \delta) < f(k_1)^\alpha$. Clearly, there is T, such that $c_t \leqslant [f(k_1) - k_1 + \beta]$ for $t \geqslant T$; also $x_t \geqslant k_1$ for all $t \geqslant 0$. Construct a sequence $\langle x', y', c'\rangle$ as follows: $x'_t = x_t$ for $t = 0, 1, \ldots, T - 1$; $x'_t = 0$ for $t \geqslant T$. Then $c'_t = c_t$ for $t = 1, \ldots, T - 1$; $c'_T = f(x_{T-1}) \geqslant f(k_1)$, $c'_t = 0$ for $t > T$. Now

$$\begin{aligned}\sum_{T=1}^{\infty} \delta^{t-1}u(c_t) &= \sum_{t=1}^{T-1} \delta^{t-1}u(c_t) + \sum_{t=T}^{\infty} \delta^{t-1}u(c_t)\\ &\leqslant \sum_{t=1}^{T-1} \delta^{t-1}u(c'_t) + \delta^{T-1}[f(k_1) - k_1 + \beta]^{\alpha}/(1-\delta)\\ &< \sum_{t=1}^{T-1} \delta^{t-1}u(c'_t) + \delta^{T-1}f(k_1)^{\alpha}\\ &\leqslant \sum_{t=1}^{T-1} \delta^{t-1}u(c'_t) + \sum_{t=T}^{\infty} \delta^{t-1}u(c'_t) = \sum_{t=1}^{\infty} \delta^{t-1}u(c'_t).\end{aligned}$$

This contradicts the optimality of $\langle x, y, c\rangle$ from $\mathbf{x} \in (k_1, \bar{k})$, and establishes our claim.

It would also be useful to have conditions on u and f, under which one could definitely say that $(x_t, y_t, c_t) \to (k_1, f(k_1), f(k_1) - k_1)$ as $t \to \infty$, for $\mathbf{x} \in [k_1, \bar{k})$. We have not found a concrete example of such a case.

Case 3. $[f'(k_1) > (1/\delta) > f'(k_2)]$.

As in Section 5a, denotes by k_*, K^* the two positive solutions of the equation $\delta f'(x) = 1$, with $k_* < K^*$. Then $k_* < k_1 < K^* < k_2$.

One qualitative property of optimal programs from $\mathbf{x} \in (0, \bar{k})$ is given in the following result.

PROPOSITION 5.2. *If* $\langle x, y, c\rangle$ *is an optimal program from* $\mathbf{x} \in (0, \bar{k})$, *then* $x_t \leqslant \max(K^*, \mathbf{x})$, *for* $t \geqslant 0$.

Proof. We establish this result, by separating two cases (i) $\mathbf{x} \in (0, K^*]$, and (ii) $\mathbf{x} \in (K^*, \bar{k})$.

In case (i), suppose on the contrary that $x_t > K^*$ for some t. Let S be the first period for which $x_S > K^*$. Then $x_{S-1} \leqslant K^*$. Also, $\delta f'(x_S) < \delta f'(K^*) = 1$. So by Lemma 5.2, $x_{S+1} > x_S$. Then this step can be repeated to show that $x_{t+1} > x_t$ for $t \geqslant S-1$. Since $x_t \leqslant \bar{k}$ for all t, so x_t converges to some $\hat{x}$. Clearly $\hat{x} > K^* > 0$. If $\hat{x} = \bar{k}$, then $\langle x, y, c\rangle$ is inefficient, a contradiction. Hence $\hat{x} < \bar{k}$. Then by Lemma 5.4, $\delta f'(\hat{x}) = 1$. But, clearly, $\delta f'(\hat{x}) < \delta f'(K^*) = 1$, a contradiction. Thus, $x_t \leqslant K^*$ for all $t \geqslant 0$.

In case (ii), suppose on the contrary that $x_t > \mathbf{x}$ for some t. Let S be the first period for which $x_S > \mathbf{x}$. Then $x_{S-1} \leqslant \mathbf{x}$. Also, $\delta f'(x_S) < \delta f'(\mathbf{x}) < \delta f'(K^*) = 1$. So by Lemma 5.2, $x_{S+1} > x_S$. Then this step can be repeated to show that $x_{t+1} > x_t$ for $t \geqslant S-1$. Now, follow exactly the argument used in case (i) to get a contradiction. Thus $x_t \leqslant \mathbf{x}$ for all $t \geqslant 0$.

In either case, then, $x_t \leqslant \max(K^*, \mathbf{x})$ for $t \geqslant 0$.

THEOREM 5.3. *If $\langle x, y, c\rangle$ is an optimal program from $\mathbf{x} \in (0, \bar{k})$, then it exhibits one of the following three behavior patterns*:

(1) $x_t \to 0$ *as* $t \to \infty$;

(2) $x_t \to K^*$ *as* $t \to \infty$;

(3) x_t *is oscillatory around* k_*.

Proof. If $\langle x, y, c\rangle$ is optimal from $\mathbf{x} \in (0, \bar{k})$, then it is an Euler program. Denote $(0, k_*]$ by A, $(k_*, K^*]$ by B, $(K^*, \bar{k})$ by C. Consider two cases: (a) $\mathbf{x} \in (0, K^*]$; (b) $\mathbf{x} \in (K^*, \bar{k})$. We consider case (a) first. Here, for any t, x_t belongs to A or B, by Proposition 5.3. There are then three subcases to consider:

(i) x_t is in A for a finite number of periods;

(ii) x_t is in B for a finite number of priods;

(iii) x_t is in A for a subsequence of periods and x_t is in B for a subsequence of periods.

In (i), there is T such that for $t \geqslant T$, x_t is in B. Hence for $t \geqslant T$, $\delta f'(x_t) \geqslant 1$. By Lemma 5.3, $c_{t+1} \geqslant c_t$ for $t \geqslant T$. By Lemma 5.5 $x_{t+1} \geqslant x_t$ for $t \geqslant T$. By Proposition 5.2, $x_t \leqslant K^*$ for $t \geqslant 0$. Hence x_t converges to some $\hat{x} \leqslant K^* < \bar{k}$. Clearly $\hat{x} > k_* > 0$ so by Lemma 5.4, $\delta f'(\hat{x}) = 1$; that is $\hat{x} = K^*$ as $t \to \infty$.

In (ii), there is T such that for $t \geqslant T$, x_t is in A. Hence for $t \geqslant T$, $\delta f'(x_t) \leqslant 1$. By Lemma 5.2, $c_{t+1} \leqslant c_t$ for $t \geqslant T$. By Lemma 5.5, $x_{t+1} \leqslant x_t$ for $t \geqslant T$. Since $x_t \leqslant a$ for $t \geqslant T$, so x_t converges to some $\hat{x} \leqslant k_*$. If $\hat{x} = 0$, then $x_t \to 0$ as $t \to \infty$. If $\hat{x} > 0$, then $\delta f'(\hat{x}) = 1$, and $\hat{x} = k_*$. Since x_t is in A for $t \geqslant T$, and $x_{t+1} \leqslant x_t$ for $t \geqslant T$, so $x_t = k_*$ for $t \geqslant T$. Hence x_t is oscillatory around a.

In (iii), by definition, x_t is oscillatory around a.

Next consider case (b); that is $\mathbf{x} \in (K^*, \bar{k})$. We consider two subcases (i′) x_t belongs to C for all $t \geqslant 0$; (ii′) x_t does not belong to C for some t. In case (i′), $\delta f'(x_t) \leqslant 1$ for all $t \geqslant 1$, so by Lemma 5.2, $c_t \geqslant c_{t+1}$ for all t, and by Lemma 5.5, $x_{t+1} \leqslant x_t$ for all t. Hence x_t converges to some $\hat{x}$. Clearly $K^* \leqslant \hat{x} \leqslant \mathbf{x} < \bar{k}$. So by Lemma 5.4, $\delta f'(\hat{x}) = 1$; consequently $\hat{x} = K^*$. Thus, $x_t \to K^*$ as $t \to \infty$.

In (ii′), consider the first period, S, for which x_S is not in C. Then there is an optimal program $\langle x', y', c'\rangle$ from $\mathbf{x} = x_S$, with $x'_t = x_{S+t}$ for $t \geqslant 0$. By the analysis of case (a), $x'_t \to K^*$ as $t \to \infty$, or $x'_t \to 0$ as $t \to \infty$, or x'_t is oscillatory around k_*. Consequently, $x_t \to K^*$ as $t \to \infty$, or $x_t \to 0$ as $t \to \infty$, or x_t is oscillatory around k_*.

Thus in both cases (a), and (b), x_t exhibits one of the three types of behavior claimed in the Theorem. ∎

If we consider the utility function given in Case 3 of Section 5a, then $x_t = K^*$ for $t \geqslant 0$ is not optimal. Following the argument of Case 2 of Section 5b, programs for which x_t converges to K^* as $t \to \infty$ are also not optimal. Thus, for that example, the behavior pattern (2) in Theorem 5.3 can be ruled out. It would be interesting to have examples in which all three types of behavior of Theorem 5.3 are actually demonstrated by optimal programs.

Case 4. $[f'(k_2) \geqslant (1/\delta) > f'(0)]$.

Dzfine, as in Section 5a, $\varnothing(x) = f(x)$ for $x \geqslant k_2$; $\varnothing(x) = [f(k_2)/k_2]x$ for $0 \leqslant x \leqslant k_2$. Then $\varnothing(0) = 0$, $\varnothing$ is strictly increasing, concave, differentiable, and satisfies the end-point conditions: $\varnothing'(\infty) < 1 < \varnothing'(0)$. Define the "expanded convex technology set" $\mathscr{T}' = \{(x, y)\colon x \geqslant 0,\ 0 \leqslant y \leqslant \varnothing(x)\}$. The non-convex technology set $\mathscr{T}$ is, of course, given by $\{(x, y)\colon x \geqslant 0,$ $0 \leqslant y \leqslant f(x)\}$. A *feasible program* $\langle x, y, c\rangle$ *for* $\mathscr{T}'$ from $\mathbf{x} > 0$, is a sequence such that

$$x_0 = \mathbf{x},\ 0 \leqslant x_t \leqslant y_t,\ y_t = \varnothing(x_{t-1}) \qquad \text{for} \quad t \geqslant 0,$$

$$0 \leqslant c_t \leqslant y_t - x_t \qquad \text{for} \quad t \geqslant 1.$$

Note that given any feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$, there is a feasible program $\langle x', y', c'\rangle$ for $\mathscr{T}'$ from $\mathbf{x}$, such that $x'_t = x_t$, $y'_{t+1} \geqslant y_{t+1}$, $c'_{t+1} \geqslant c_{t+1}$ for $t \geqslant 0$. Also $\langle x, y, c\rangle$ itself is a feasible program for $\mathscr{T}'$ from $\mathbf{x}$.

An *optimal program* $\langle x^*, y^*, c^*\rangle$ *for* $\mathscr{T}'$ from $\mathbf{x} > 0$ is a feasible program for $\mathscr{T}'$ such that

$$\limsup_{t \to \infty} \sum_{t=1}^{T} \delta^{t-1}[u(c_t) - u(c_t^*)] \leqslant 0$$

for every feasible program $\langle x, y, c\rangle$ for $\mathscr{T}'$ from $\mathbf{x}$.

As in Section 5a, let k_*, K^* be the two positive solutions of $\delta f'(x) = 1$, with $k_* < K^*$. Then $k_* < k_1 < k_2 \leqslant K^*$.

The qualitative properties of optimal programs from $\mathbf{x} \in (k_2, \bar{k})$ is given in the following Theorem.

THEOREM 5.4. (1) *If* $\langle x, y, c\rangle$ *is an optimal program from* $\mathbf{x} \in [k_2, K^*]$, *then* (x_t, y_t, c_t) *are monotonically non-decreasing, and* $(x_t, y_t, c_t) \to (K^*, f(K^*), f(K^*) - K^*)$ *as* $t \to \infty$;

(2) *If* $\langle x, y, c\rangle$ *is an optimal program from* $\mathbf{x} \in (K^*, \bar{k})$, *then* (x_t, y_t, c_t) *are monotonically non-increasing, and* $(x_t, y_t, c_t) \to (K^*, f(K^*),$ $f(K^*) - K^*)$ *as* $t \to \infty$.

Proof. To prove (1), consider an optimal program $\langle x', y', c' \rangle$ for $\mathscr{E}'$, from $\mathbf{x} \in [k_2, K^*]$; there exists one by Majumdar (1975, Theorem 1), and is unique since $\varnothing$ is concave and u strictly concave. It is standard to check [follow, for example, the analysis in Mitra (1979, Section 5)] that (x_t', y_t', c_t') are monotonically non-decreasing, and $(x_t', y_t', c_t') \to (K^*, f(K^*), f(K^*) - K^*)$ as $t \to \infty$. Since $x_t' \in [k_2, K^*]$ for all t, hence $\langle x', y', c' \rangle$ is a feasible program from $\mathbf{x} \in [k_2, K^*]$. Hence $\langle x', y', c' \rangle$ is a feasible program from $\mathbf{x} \in [k_2, K^*]$. Hence $\langle x', y', c' \rangle$ is a feasible program from $\mathbf{x} \in [k_2, K^*]$.

We claim that if $\langle x, y, c \rangle$ is an optimal program from $\mathbf{x} \in [k_2, K^*]$, then $(x_t, y_{t+1}, c_{t+1}) = (x_t', y_{t+1}', c_{t+1}')$ for $t \geqslant 0$. Otherwise, if $(x_t, y_{t+1}, c_{t+1}) \neq (x_t', y_{t+1}', c_{t+1})$ for some t, then since $\langle x, y, c \rangle$ is a feasible program for $\mathscr{E}'$, so $\sum_{t=1}^{\infty} \delta^{t+1} u(c_t) < \sum_{t=1}^{\infty} \delta^{t-1} u(c_t')$, as $\langle x', y', c' \rangle$ is the *unique* optimal program for $\mathscr{E}'$. So $\langle x, y, c \rangle$ could not be an optimal program from $\mathbf{x} \in [k_2, K^*]$, a contradiction, which establishes our claim.

Thus if $\langle x, y, c \rangle$ is an optimal program from $\mathbf{x} \in [k_2, K^*]$, then $\langle x, y, c \rangle = \langle x', y', c' \rangle$ and so (x_t, y_t, c_t) are monotonically non-decreasing, and $(x_t, y_t, c_t) \to (K^*, f(K^*), f(K^*) - K^*)$ as $t \to \infty$.

One can prove (2) by analogous arguments, so we omit the details.

Remark. Note that in the process of proving Theorem 5.4, we have also proved that there is a unique optimal program from every $\mathbf{x} \in [k_2, \bar{k})$.

The behavior of optimal programs from $\mathbf{x} \in (0, k_2)$ can be obtained by checking that the analysis in Proposition 5.2 and Theorem 5.3 can be applied to this case.

THEOREM 5.5. *If $\langle x, y, c \rangle$ is an optimal program from* $\mathbf{x} \in (0, k_2)$, *then $x_t \leqslant K^*$ for $t \geqslant 0$. Furthermore, it exibits one of the following three behavior patterns*: (1) $x_t \to 0$ *as* $t \to \infty$; (2) $x_t \to K^*$ *as* $t \to \infty$; (3) x_t *is oscillatory around* k_*.

Proof. Follow the proofs of Proposition 5.2, and Theorem 5.3. ∎

We have seen in Cases 2, 3, and 4 that optimal programs from different initial stocks may exhibit quite different types of behavior, even in the longrun. In the "classical" model, this possibility cannot occur. Thus, this can be considered as an important characteristic of the "non-classical" model. However, we have not yet given any concrete example, to demonstrate that optimal programs from different initial stocks *will*, in fact, have different long-run behavior. Case 4 is a convenient case to construct such an example. We know that (irrespective of the choice of a utility function) if $f'(k_2) = (1/\delta)$, then the feasible program $\langle x, y, c \rangle$ from k_2 with $x_t = k_2$ for $t \geqslant 0$, is an optimal program from k_2. We will construct a utility function such that an optimal program $\langle x', y', c'\rangle$ from a "sufficiently low" input level $\mathbf{x} > 0$, will remain bounded away from k_2.

EXAMPLE. Assume $f'(k_2) = (1/\delta)$. Define $k_3 = (k_2/2)$. Choose $k_4 > 0$, such that $f[f(k_4)] = \min[\frac{1}{2}, k_3]$. Define $\beta = \frac{1}{2}f(k_3)[1 - \{\delta f(k_3)/k_3\}]$. Note that $k_4 < k_2$, so $\{\delta f(k_4)/k_4\} < 1$. Choose $\frac{1}{2} < \alpha < 1$ such that $[1 + (1/\delta)][(1-\alpha)/(1-\delta)] < (\beta/2)$. Define $u(c) = c^{\alpha}$.

Consider, now, an optimal program $\langle x', y', c'\rangle$ from $\mathbf{x} = k_4$. We claim that x_t' is bounded away from k_2. Suppose, on the contrary that there is a subsequence of periods for which $x_t' \to k_2$. By the argument used to establish Proposition 5.3, $x_t' \leqslant k_2$ for all t. Thus, there is some t, such that $k_2 - x_t' \leqslant k_2 - f[f(k_4)]$. Let S be the first period this happens. Then $x_S' \geqslant f[f(k_4)]$. Hence $S \geqslant 2$. Since S is the first period for which this happens, so $k_2 - x_{S-1}' > k_2 - f[f(k_4)]$. So $x_{S-1}' < f[f(k_4)]$, and similarly $x_{S-2}' < f[f(k_4)]$. Also $x_S' \leqslant f(x_{S-1}')$, so $x_{S-1}' \geqslant f^{-1}[x_S'] \geqslant f(k_4)$; $x_{S-1}' \leqslant f(x_{S-2}')$, so $x_{S-2}' \geqslant f^{-1}[x_{S-1}'] \geqslant k_4$.

Define a sequence $\langle x'', y'', c''\rangle$ from $\mathbf{x}$ as follows: $x_t'' = x_t'$ for $t = 0, \ldots, S-2$; $x_t'' = 0$ for $t \geqslant S-1$. Clearly, this is a feasible program from $\mathbf{x}$. Now $\delta^{t-1}u(c_t'') = \delta^{t-1}u(c_t')$ for $t = 1, \ldots, S-2$. For $t = S-1$, $\delta^{t-1}u(c_t'') = \delta^{t-1}c_t''^{\alpha} = \delta^{t-1}f(x_{t-1}'')^{\alpha} = \delta^{t-1}f(x_{t-1}')^{\alpha}$; for $t > S-1$, $\delta^{t-1}u(c_t'') = 0$.

For $t \geqslant S$, we have

$$\begin{aligned}
\delta^{t-1}u(c_t') &= \delta^{t-1}c_t'^{\alpha} \leqslant \delta^{t-1}\alpha c_t' + \delta^{t-1}(1-\alpha) \\
&= \delta^{t-1}\alpha f(x_{t-1}') - \delta^{t-1}\alpha x_t' + \delta^{t-1}(1-\alpha) \\
&= \left\{\delta^{t-2}\alpha\left[\frac{\delta f(x_{t-1}')}{x_{t-1}'}\right]x_{t-1}' - \delta^{t-2}\alpha x_{t-1}'\right\} \\
&\quad + \delta^{t-2}\alpha x_{t-1}' - \delta^{t-1}\alpha x_t' + \delta^{t-1}(1-\alpha) \\
&= -\mu_{t-2} + \delta^{t-2}\alpha x_{t-1}' - \delta^{t-1}\alpha x_t' + \delta^{t-1}(1-\alpha).
\end{aligned}$$

Now, $\mu_{t-2} = \delta^{t-2}\alpha x_{t-1}'[1 - \{\delta f(x_{t-1}')/x_{t-1}'\}] \geqslant 0$ for $t \geqslant S$. For $t = S$, we have

$$\begin{aligned}
&\delta^{t-2}\alpha x_{t-1}'[1 - \{\delta f(x_{t-1}')/x_{t-1}'\}] \\
&\qquad \geqslant \frac{\delta^{t-2}}{2} f(k_4)\left[1 - \frac{\delta f(k_3)}{k_3}\right] \geqslant \delta^{t-2}\beta.
\end{aligned}$$

Also, for $t = S-1$,

$$\delta^{t-1}u(c_t') = \delta^{t-1}c_t'^{\alpha} \leqslant \delta^{t-1}\alpha f(x_{t-1}') - \delta^{t-1}\alpha x_t' + \delta^{t-1}(1-\alpha).$$

So

$$\sum_{t=S}^{\infty} \delta^{t-1} u(c'_t) = \sum_{t=S}^{\infty} (\delta^{t-2} a x'_{t-1} - \delta^{t-1} a x'_t) - \sum_{t=S}^{\infty} \mu_{t-2} + \sum_{t=S}^{\infty} \delta^{t-1}(1-\alpha)$$
$$\leqslant \delta^{S-2} \alpha x'_{S-1} - \delta^{S-2}\beta + \delta^{S-1} \frac{(1-\alpha)}{(1-\delta)}.$$

Hence,

$$\sum_{t=S-1}^{\infty} \delta^{t-1} u(c'_t) \leqslant \delta^{S-2} af(x'_{S-2}) - \delta^{S-2}\beta + \delta^{S-2}(1-\alpha) + \delta^{S-1} \frac{(1-\alpha)}{(1-\delta)}$$
$$\leqslant \delta^{S-2} af(x'_{S-2}) - \delta^{S-2}\beta + \delta^{S-2}(1-\alpha) \frac{[1+(1/\delta)]}{(1-\delta)}$$
$$< \delta^{S-2} af(x'_{S-1}).$$

So

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c'_t) < \sum_{t=1}^{S-2} \delta^{t-1} u(c'_t) + \delta^{S-2} af(x'_{S-2})$$
$$= \sum_{t=1}^{S-2} \delta^{t-1} u(c''_t) + \delta^{S-2} af(x'_{S-2})$$
$$< \sum_{t=1}^{S-2} \delta^{t-1} u(c''_t) + \delta^{S-2} f(x'_{S-2})$$
$$< \sum_{t=1}^{S-2} \delta^{t-1} u(c''_t) + \delta^{S-2} [f(x'_{S-2})]^{a}$$
$$= \sum_{t=1}^{S-2} \delta^{t-1} u(c''_t) + \sum_{t=S-1}^{\infty} \delta^{t-1} u(c''_t) = \sum_{t=1}^{\infty} \delta^{t-1} u(c''_t).$$

Hence $\langle x', y', c' \rangle$ is not optimal from **x**, a contradiction. This establishes our claim that the optimal program $\langle x', y', c' \rangle$ remains bounded away from k_2 in input levels. Clearly, the feasible program $\langle x, y, c \rangle$ from k_2, given by $x_t = k_2$ for all t is an optimal program from k_2 (by the analysis of Case 4 in Section 5a). Hence optimal programs from k_2 and **x** exhibit different long-run behavior.

Case 5. $[f'(0) \geqslant (1/\delta)]$.
As in Section 5a, call K^* the unique positive solution of $\delta f'(x) = 1$. Then

$k_2 < K^* < k^*$. The qualitative behavior of optimal programs from $\mathbf{x} \in (0, \bar{k})$ is given by the following result:

PROPOSITION 5.4. *If $\langle x, y, c\rangle$ is an optimal program from $\mathbf{x} \in (0, \bar{k})$, then*

(1) $\mathbf{x} \leqslant K^*$ *implies* $x_t \leqslant K^*$, $x_{t+1} \geqslant x_t$, *and* $c_{t+2} \geqslant c_{t+1}$ *for all* $t \geqslant 0$.

(2) $\mathbf{x} \geqslant K^*$ *implies* $x_t \geqslant K^*$, $x_{t+1} \leqslant x_t$, *and* $c_{t+2} \leqslant c_{t+1}$ *for all* $t \geqslant 0$.

Proof. We will first prove (1). We claim that $x_t \leqslant K^*$ for $t \geqslant 0$. Otherwise, let S be the first period for which $x_S > K^*$ for $t \geqslant 0$. Then $x_{S-1} \leqslant K^*$, so $x_S > x_{S-1}$. Also $\delta f'(x_S) < 1$. So by Lemma 5.2, $c_{S+1} < c_S$, $x_{S+1} > x_S$. Repeating this step $x_{t+1} > x_t$ and $c_{t+1} < c_t$ for $t \geqslant S$. This contradicts Lemma 5.5. Hence $x_t \leqslant K^*$ for $t \geqslant 0$. So $\delta f'(x_t) \geqslant 1$ for $t \geqslant 0$, and by Lemma 5.3, $c_{t+2} \geqslant c_{t+1}$ for $t \geqslant 0$. Finally, by Lemma 5.5, $x_{t+1} \geqslant x_t$ for $t \geqslant 0$.

Next, we prove (2). We claim that $x_t \geqslant K^*$ for $t \geqslant 0$. Otherwise, let S be the first period for which $x_S < K^*$. Then $x_{S-1} \geqslant K^*$, so $x_S < x_{S-1}$. Also, $\delta f'(x_S) > 1$. So by Lemma 5.3, $c_{S+1} > c_S$, $x_{S+1} < x_S$. Repeating this step, $x_{t+1} < x_t$ and $c_{t+1} > c_t$ for $t \geqslant S$. This contradicts Lemma 5.5. Hence, $x_t \geqslant K^*$ for $t \geqslant 0$. So $\delta f'(x_t) \leqslant 1$ for $t \geqslant 0$, and by Lemma 5.2, $c_{t+2} \leqslant c_{t+1}$ for $t \geqslant 0$. Finally, by Lemma 5.5, $x_{t+1} \leqslant x_t$ for $t \geqslant 0$.

The following asymptotic behavior of optimal programs is then easy to obtain.

THEOREM 5.5. *If $\langle x, y, c\rangle$ is an optimal program from $\mathbf{x} \in (0, \bar{k})$, then $(x_t, y_t, c_t) \to (K^*, f(K^*), f(K^*) - K^*)$ as $t \to \infty$.*

Proof. If $\mathbf{x} \in (0, K^*]$, then by Proposition 5.4, $x_t \to \hat{x}$ as $t \to \infty$, where $\mathbf{x} \leqslant \hat{x} \leqslant K^*$. So by Lemma 5.4, $\delta f'(\hat{x}) = 1$; that is, $\hat{x} = K^*$.

If $\mathbf{x} \in [K^*, \bar{k})$, then by Proposition 5.4, $x_t \to \tilde{x}$ as $t \to \infty$, where $K^* \leqslant \tilde{x} \leqslant \mathbf{x}$. So by Lemma 5.4, $\delta f'(\tilde{x}) = 1$; that is, $\tilde{x} = K^*$.

Thus $x_t \to K^*$ as $t \to \infty$ for all $\mathbf{x} \in (0, \bar{k})$. So $(y_t, c_t) \to (f(K^*), f(K^*) - K^*)$ as $t \to \infty$.

6. DISCOUNTED OPTIMAL GROWTH WITH A LINEAR UTILITY FUNCTION

We shall sketch the main results that one can obtain if one maintains the technological structure of the model (described in Sections 2b and 2c) but takes a linear utility function $u(c) = c$. We recast (2.4) as follows: a feasible program $\langle x^*, y^*, c^*\rangle$ from $\mathbf{x}$ is optimal if:

$$\sum_{t=1}^{\infty} \delta^{t-1} c_t \leqslant \sum_{t=1}^{\infty} \delta^{t-1} c_t^* \tag{6.1}$$

for every feasible program $\langle x, y, c\rangle$ from $\mathbf{x}$. Note right away that an optimal program from $\mathbf{x} > 0$ need not be a positive program. Thus, a mechanical application of many of the arguments in Section 5 is not admissible. The existence question is settled by the direct compactness argument in Majumdar [1975, Theorem 1]. Turning to the question of characterization of optimal programs, it is useful to single out two special programs. The program $\langle x, y, c\rangle$ from $\mathbf{x} > 0$ defined as

$$c_1 = y_1 = f(\mathbf{x}), x_t = y_{t+1} = c_{t+1} = 0 \qquad \text{for} \quad t \geqslant 1, \tag{6.2}$$

is called the *extinction program* from $\mathbf{x} > 0$. Of course, the extinction program is dissipative. The program $\langle \bar{x}, \bar{y}, \bar{c}\rangle$ from $\mathbf{x} > 0$ defined as

$$\bar{x}_0 = \mathbf{x}; \bar{x}_t = \bar{y}_t = f(\bar{x}_{t-1}), \bar{c}_t = 0 \qquad \text{for} \quad t \geqslant 1. \tag{6.1}$$

is called the *pure accumulation program* from $\mathbf{x} > 0$. As before, the equation $\delta f'(x) = 1$ is important in the analysis of qualitative behavior of optimal programs. It may not have a real non-negative solution. If it has a unique non-negative real solution, we denote it by K^*. If it has two non-negative real solutions, the larger solution is denoted by K^*.

In contrast with the analysis of Section 5, the results with a linear utility function are sharper and more complete. One considers three distinct possibilities:

Case 1. $\delta f'(k_2) \leqslant 1$.

Case 2. $\delta f'(0) < 1 < \delta f'(k_2)$.

Case 3. $\delta f'(0) \geqslant 1$.

In case 1, the extinction program (defined in (6.2)) is optimal from every initial stock $\mathbf{x} > 0$. If, in fact, $\delta f'(k_2) < 1$ the extinction program is also the unique optimal program.

In case 3, if $\mathbf{x} \geqslant K^*$, the program defined as $x_0^* = \mathbf{x}$, $x_t^* = K^*$ for $t \geqslant 1$ is the unique optimal program. On the other hand, for $\mathbf{x} < K^*$, let M be the smallest integer such that $\bar{x}_M \geqslant K^*$, i.e., let M be the first period such that the pure accumulation program from $\mathbf{x}$ (defined in (6.3)) has an input stock at least as large as K^*. If $\mathbf{x} < K^*$, the feasible program given by $x_t^* = \bar{x}_t$ for $t = 0,..., M-1$, $x_t^* = K^*$ for $t \geqslant M$ is the unique optimal program. In other words, the optimal program coincides with the pure accumulation program in the initial periods till one "attains" K^*, and then uses K^* as the input in all subsequent periods. Thus, we have the usual "turnpike behavior" of optimal programs: independent of initial stock $\mathbf{x} > 0$, the stationary input K^* and the stationary consumption $f(K^*) - K^*$ is attained in a finite number of periods.

Case 2 is perhaps the central case, since it illustrates a basic difference between the "classical" and the "non-classical" models. In this case, one is able to establish a critical "threshold level", $\mathbf{k}(0 < \mathbf{k} < \bar{k})$ such that if the initial stock $\mathbf{x}$ is less than $\mathbf{k}$, the extinction program is optimal from $\mathbf{x}$. On the other hand, if $\mathbf{x}$ is greater than $\mathbf{k}$, there is $M' \geqslant 1$, such that an optimal program $\langle x, y, c \rangle$ satisfies $x_t = K^*$ for $t \geqslant M'$. Thus, the behavior of optimal programs depends critically on the magnitude of the initial stock.

Cases 1 and 3 were considered by Clark (1971). His conjecture about the existence of $\mathbf{k}$ in case 2 is proved in Majumdar and Mitra (1980) which has a self-contained discussion of all the three cases.

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